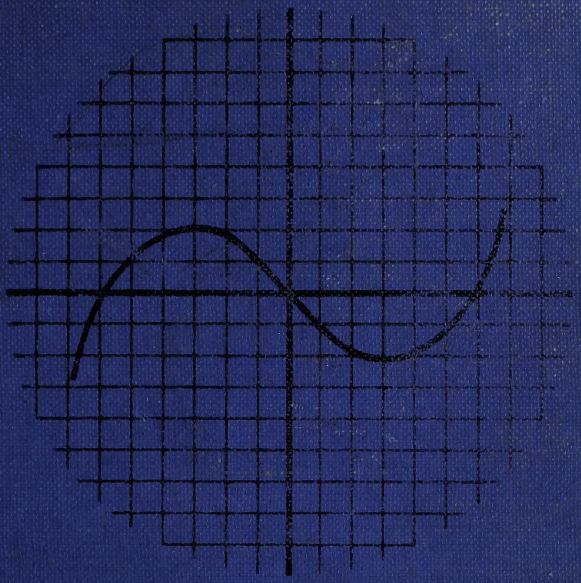


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SECOND COURSE



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ALGEBRA FOR TODAY

Second Course

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PREFACE

"Algebra for Today, Second Course," like "Mathematics for Today," owes its origin to the conviction that elementary mathematics can and should be made more meaningful to young pupils, and that there is no real conflict between the insistent new demands in education and the views of progressive thinkers in the field of mathematics. Throughout the preparation of this text no effort has been spared to give greater prominence to those dominant ideas, attitudes, and types of thinking which are now generally regarded as the essential contribution of elementary algebra in any worth-while scheme of secondary education.

In Canadian schools this subject is generally regarded as compulsory for university matriculation requirements and is offered in one or more years of the senior high school. To meet varying school practices, it is obviously necessary either to provide a corresponding variety of courses of study and of text or to create a single course which shall have the requisite flexibility and can therefore be adapted to the needs of a particular school or group of pupils. The latter plan is the one which has been embodied in this text. Certain topics and exercises are designated as optional, and may be assigned at the discretion of the teacher to such of the students as are able to undertake them.

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Among the significant features in the development of the programme incorporated in this text, attention may be called to the following:

1. The Introduction and the first chapter aim to arouse an *immediate interest* in the subject, and offer a *clear preview of the purpose of algebra*. A consistent *motivation* marks the presentation of all new topics.

2. The entire course is consciously built around *three major types of work*, namely, (1) *the essential technique of algebra*; (2) the functional *core of algebra, comprising the formula, the equation, and the graph*; and (3) *the solution of problems*. A clear recognition of these three main fields of emphasis has led to greater perspective and to a desirable concentration on the essentials.

3. The *unit plan*, followed throughout, contributes materially to definiteness and to a considerable economy of effort.

4. In developing the *technique of algebra* the authors have aimed at (1) *insight* rather than mechanical imitation; (2) *comprehensiveness*; (3) *thoroughness*; and (4) *saneness*. Types of exercises which recent progressive syllabi have uniformly designated as obsolete or undesirable have either been omitted entirely or have received a very much reduced degree of emphasis.

5. *The functional programme of algebra*, in its threefold aspect, is stressed from the very beginning and is made the *central theme of the entire course* (see, especially, Chapters I, II, V, VII). At the same time, unnecessary technical details or involved definitions have been avoided.

6. The *solution of problems* receives careful attention.

7. *Graphic methods*, which are employed consistently whenever they serve a useful purpose, are treated as a basic tool and not as mere incidentals.

8. An unusual amount of attention is given to *worthwhile applications* within the comprehension of the learner.

9. Significant *illustrations, quotations* from eminent mathematicians and educators, and *historical notes,* which abound throughout the text, are intended to stress the *cultural importance of mathematics.*

10. The *explanations and necessary theoretical discussions* are not only unusually complete, but aim at the utmost *simplicity consistent with scientific accuracy.*

11. The *chapter tests,* supplemented by numerous sets of *review exercises,* make it possible to *diagnose the pupil's progress* in the mastery of each of the principal topics.

12. The *general summary and review* at the end of the book is based on the three main fields of emphasis mentioned above, thus giving a very comprehensive picture of the entire course.

In the preparation of this text the authors have depended not only on their own extensive classroom experience in this field of work, but also on the constant counsel and co-operation of outstanding teachers and specialists. They have kept before them "A Report Prepared for the Mathematical Association: The Teaching of Algebra in Schools (G. Bell and Sons, Ltd., London, 1934) and acknowledge their indebtedness to the findings of the committee. The authors wish to express their deep appreciation to all those who have assisted them either directly or indirectly.

The courteous permissions of these publishers for copyrighted material are gratefully acknowledged: Thornton Butterworth, Ltd., for a quotation from A. N. Whitehead's "Introduction to Mathematics"; Longmans, Green and Co., for quotations from Sir T. Percy Nunn's "Teaching of Algebra."

THE AUTHORS

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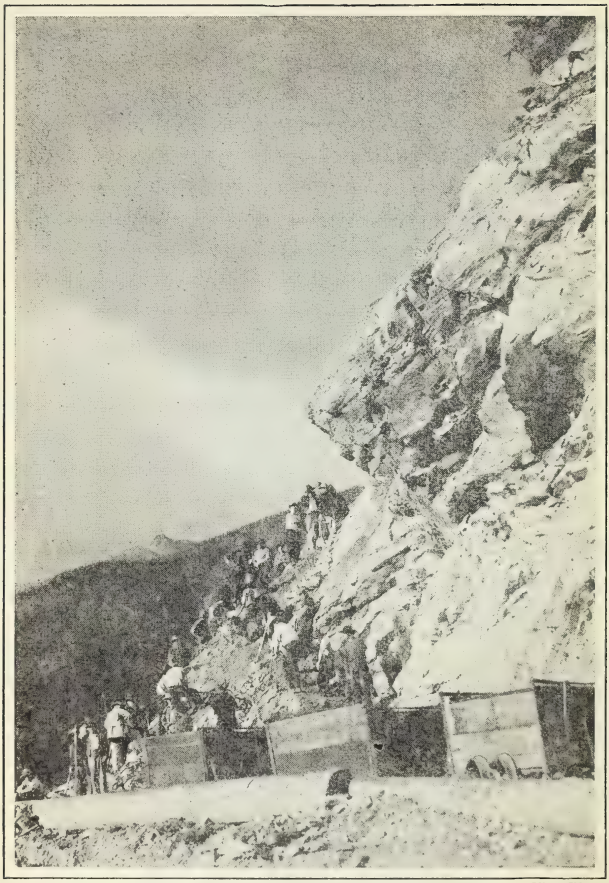
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Wide World

The building of the Canadian Pacific Railway

Algebraic formulas were essential in the building of this railway. This scene is from a motion picture filmed in British Columbia

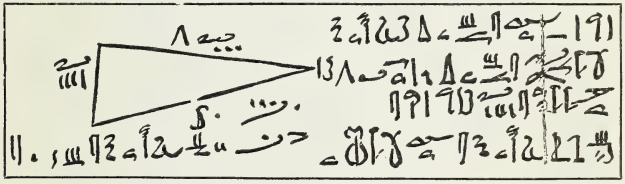
ALGEBRA FOR TODAY

Second Course

INTRODUCTION

How Mathematics Developed. As you know, mathematics is a very old science. Its beginnings go back to the dawn of human history. It owes its origin to the fact that at all times people found it necessary to count and to measure.

In particular, farming and building led to the discovery of the most common rules of mensuration. Historical records prove that in ancient Egypt and in Babylonia,



Egyptian land measure, from the Ahmes Papyrus

The manuscript, of which a small portion is shown above, was written by the Egyptian scribe Ahmes about 1650 B.C. The papyrus was originally a single roll about 18 feet long and 13 inches high. It is now in the possession of the British Museum. Reading from right to left, the translation of the above passage is as follows: "Example of making a triangle in land: If it is said to thee, A triangle of *khet* 10 on the side of it, *khet* 4 in the base of it, what is the area of it? The doing as it occurs:

1	400
$\frac{1}{2}$	200
1	1,000
2	2,000

The area of it this is: 20 *setat*"

thousands of years ago, the surveyors and architects were acquainted with the fundamental facts of arithmetic and geometry.

This early knowledge was gradually extended and perfected. In fact, mathematics is the joint creation of the ages, the common heritage of all mankind. The story of the development of mathematics in India, Greece, Arabia, Italy, and northern Europe is of absorbing interest. To this day mathematics has continued to be a very live, constantly growing science.

The Shorthand of Mathematics. Mathematics may be compared to an old tree. It has many roots and branches. Algebra is one of these branches. Like arithmetic and geometry, it developed slowly. In its present form, however, it is only a few hundred years old.

Algebra is often described as *mathematical shorthand*, which may be used to write formulas and to solve numerical problems quickly and accurately. *A mathematical formula arises when a numerical rule is written in symbols.* That is, instead of writing such a rule in the usual verbal form, we may shorten the important mathematical words appearing in the rule by using initials or other convenient letters, and connecting them by certain signs that indicate the relations expressed by the rule.

Thus, we may shorten the rule "*The area of a triangle is equal to half the product of its base and its height*" by writing the formula $A = \frac{1}{2}\, bh$.

There are many such rules. It is one of the chief purposes of algebra to furnish methods of shortening these rules and of applying the resulting formulas. In this manner algebra has become one of the chief economy-devices of our age.

But aside from this practical or technical use, the shorthand of mathematics serves a higher purpose. The invention of symbols has been called the greatest single event in human history. Without symbols there could be no language, no real thinking, and we should always remain the slaves of *things*.

Take, for example, any spoken or written word such as "tree." A child learns what "tree" means only after seeing many trees and hearing the word "tree" used correctly again and again. But when he has once mastered the meaning of this word, he can use it in his speech or in his reading without being obliged to point out an actual tree. That is, the word "tree" has become a substitute for *any* tree. It has become a *symbol*. In like manner, all spoken or written words, whether they refer to objects or ideas, are symbols which enable us to communicate quickly with others, by a sort of magic.

The Central Theme of Algebra. To get a clear idea as to the purpose of any school subject, such as algebra, it is helpful to look at its general features very much as an observer in an airplane looks at a landscape. In such a general view, what is the outstanding message, or "central theme" of algebra? The answer is that *algebra is largely concerned with a study of relationships*.

Everyone knows, for example, that the prosperity of Canada depends on such factors as abundant crops, good wages, the total output of industrial plants, building activities, transportation, the volume of exports and imports, and the like. Evidently all these factors are more or less interrelated. It is therefore a matter of very great importance, for the welfare of the country and of each individual citizen, that these relationships be investigated.

For if it can be shown that definite laws govern such relationships, it may be possible to *control* them. This is exactly what happens when great industrial leaders and business men make co-operative plans for the improvement of economic conditions.

Now, this illustration may serve to remind us of the *tremendously important fact that we are living in a scientific age and that scientific thinking is replacing a policy of trusting to chance.* Hence we find that in countless laboratories the world's scientists, engineers, and inventors are investigating the secrets of nature with a view to improving the conditions on which human life depends. Modern science has shown that there is a regularity, or rhythm, in the changes which are constantly going on all about us. There are definite laws which control the giant suns in distant regions of the universe no less than the motions of the tiniest particle of matter. By understanding and mastering these laws, we may hope to deal more intelligently with the world in which we live. This is true especially if these laws can be expressed in a concise way that is generally understood.

Algebra has contributed the machinery for expressing in a universal language the important relationships which have become the backbone of science, engineering, and industry. In particular, the three great tools which algebra has furnished for this purpose are the *formula*, the *equation*, and the *graph*.

In this book we shall therefore give constant attention to the study of relationships, and we shall build the details of the subject around its three central topics.

CHAPTER I

FUNDAMENTAL LAWS AND OPERATIONS

1. The Operations of Algebra

1. Important Relations of the Operations. In arithmetic we constantly use the "four fundamental operations," namely, addition, subtraction, multiplication, and division. Besides these four operations, algebra has occasion to use three others. These may be described as the operation of finding a *power* (*involution*), the operation of finding a *root* (*evolution*), and the operation of finding an *exponent* (*logarithmation*). The relations of these *seven* operations to each other will now be considered.

1. Compare the processes of *addition* and *subtraction* in the illustrations below:

Addition		Subtraction	
3	Addend	7	Minuend
4	Addend	4	Subtrahend
7	Sum	3	Difference

Observe that the *minuend* 7 corresponds to the *sum* 7, while the *subtrahend* 4 and the *difference* 3 correspond to the *addends* 4 and 3. Hence we have the important relation

Subtrahend + *difference* = *minuend.*

Evidently the process of subtraction "undoes" a corresponding process of addition. Thus, $5 + 3 - 3 = 5$, and $a + b - b = a$. That is, *subtraction is the inverse of addition.*

2. Now, the addends to be combined may all be alike. In that case, the process of addition may be shortened by the introduction of a new operation, namely, *multiplication.*

Thus, instead of writing $4 + 4 + 4 = 12$, we may write $3 \times 4 = 12$. Similarly, we may replace $a + a + a$ by $3 \times a$, or $3a$.

Compare the processes of *multiplication* and *division* in the following illustrations:

Multiplication	Division
4 Multiplicand (Factor) 3 Multiplier (Factor) 12 Product	$\frac{12 \text{ Dividend}}{3 \text{ Divisor}} = 4$ Quotient

Observe that the *dividend* 12 corresponds to the *product* 12, while the *divisor* 3 and the *quotient* 4 correspond to the *factors* 3 and 4 of the *product* 12. Hence we have the important relation

$$\textit{Divisor} \times \textit{quotient} = \textit{dividend}.$$

It is evident that the process of division "undoes" a corresponding process of multiplication. Thus,

$$(8 \times 4) \div 4 = 8,$$

and

$$(a \times b) \div b = a.$$

That is, *division is the inverse of multiplication.*

3. Just as we may have like addends, we may have like factors. In that case, we find it convenient to use a new operation, namely, *involution.*

Thus, instead of writing $2 \times 2 \times 2 = 8$, we may write $2^3 = 8$. Here the number 2 is called the *base,* 3 is called the *exponent,* and 8 is called the *power.* Similarly, 3^4 means $3 \times 3 \times 3 \times 3$, or 81. Observe that in such a case the exponent shows how many times the base is repeated as a

factor. The expression 2^3 is read "2 *cube*" or "2 to the third power," while 2^4 is read "2 to the fourth power," and so on.*

Now observe that the relation $2^3 = 8$ suggests three possible problems, each of which requires for its solution a special operation. Thus we may have

(1) $2^3 = (\ )$, [Involution]
or (2) $(\ )^3 = 8$, [Evolution]
or (3) $2^{(\ )} = 8$. [Logarithmation]

In the first case the *power* is required. The process necessary to find the power is called *involution*. In the second case the *base* is required. Here we apply a process that is called *evolution*. In the third case the *exponent* is required. This problem leads to a process called *logarithmation*.

The relation between *involution* and *evolution* may be shown as follows:

Involution	Evolution
$2 \times 2 \times 2 = (\ )$,	$(\ ) \times (\ ) \times (\ ) = 8$,
or $2^3 = (\ )$.	or $(\ )^3 = 8$.†

In the above illustration of evolution, the number 8 is called the *radicand*, the exponent 3 is called the *index*, while the required base is called the *root*. That is, the radicand corresponds to the *power*, while the root corresponds to the *base*. Hence we have the important relation

Root raised to power shown by index = radicand.

* In like manner, a means a^1, and is read "a to the first power"; while a^2 means $a \times a$, and is read "a square," or "a to the second power"; a^3 means $a \times a \times a$, and is read "a cube," or "a to the third power," and so on.

† This relation is usually indicated by writing $\sqrt[3]{8} = (\ )$. The symbol $\sqrt{\ }$ is called the *radical sign*. The horizontal bar which is generally attached to this symbol really represents parentheses. Evidently, $\sqrt[1]{a}$ has the same meaning as a; $\sqrt[2]{a}$ is usually written $\sqrt{a}$, which is read "the *square root* of a"; $\sqrt[3]{a}$ is read "the *cube root* of a," and so on.

That is, *evolution and involution are inverse processes.* Thus, $(\sqrt[n]{a})^n = a$. It will also be seen that *logarithmation* and *involution* are inverse processes.

4. We may now classify the seven operations discussed above as follows:

Direct Operations	Inverse Operations
1. Addition	1. Subtraction
2. Multiplication	2. Division
3. Involution	3. Evolution
	4. Logarithmation

2. Sequence of Numerical Operations. Addition and subtraction are often called operations of the *first order*, while multiplication and division are operations of the *second order*, and involution, evolution, and logarithmation are operations of the *third order*.

1. When several numerical operations occur in a given expression, it is customary to apply the following rule:

Perform first the indicated operations of the ***third order,*** *then those of the* ***second order,*** *and finally those of the* ***first order.***

For example,

$$\begin{aligned} 4^3 + 5 \times 3^2 - 6 \times 2^3 &= 64 + 5 \times 9 - 6 \times 8 \\ &= 64 + 45 - 48 \\ &= 61. \end{aligned}$$

According to this rule, observe that $4 + 5 \times 2$ does *not* mean 9×2, but $4 + 10$, or 14. Likewise, $16 + 8 \div 4$ does *not* mean $24 \div 4$, but $16 + 2$, or 18.

2. When an expression involves successive multiplications and divisions, it is desirable to perform the indicated operations *in the order in which they occur.* In case of doubt, the exact order should be indicated by using parentheses. In

such a case the operations indicated within a parenthesis should be performed first.

Thus, $20 \times 10 \div 2 = (20 \times 10) \div 2$, or 100. Also, according to the agreement stated above, $20 \div 10 \times 2$ should be regarded as equivalent to $(20 \div 10) \times 2$, or 4, and *not* to $20 \div (10 \times 2)$, or 1.

2. Signed Numbers

3. Definitions. A *positive* number is a number which is preceded by a plus sign; a *negative* number is a number preceded by a minus sign.

Thus, $+4$, $+\frac{1}{2}$, $+\sqrt{5}$ are positive numbers, while -3, -5.6, $-\sqrt{7}$ are negative numbers. Whenever a number is written without a sign, the plus sign is understood.

Positive and negative numbers are often called *signed* numbers or *algebraic* numbers or *directed* numbers.

To every positive number there corresponds a definite negative number. Thus, $+4$ corresponds to -4. Such numbers are said to have the same *absolute value*. That is, *the absolute value of a signed number, or its numerical value, is its value regardless of its sign.*

4. Relation of Positive and Negative Numbers. Suppose that you had a weekly income of \$60, but that your weekly expenses were also \$60. Evidently, the combined effect of having an income of \$60 and expenses of \$60 is zero. That is,

$$(+60) + (-60) = 0.$$

Illustrations of this type suggest the following general relation:

$$(+n) + (-n) = 0.$$

5. Rules of Signs. By keeping in mind the fundamental relation pointed out above, it is easy to justify certain rules of signs which are of great importance in algebra.

1. *Addition.* In the following statements, we may interpret $+5$ to mean "a gain of 5," while -5 is regarded as "a loss of 5," and so on. We then see that

$$(+5)+(+3)=+8. \qquad (+5)+(-3)=+2.$$
$$(-5)+(-3)=-8. \qquad (-5)+(+3)=-2.$$

That is, we have the following rules for the addition of signed numbers:

1. *To add two signed numbers having* ***like*** *signs, find the* ***sum*** *of their absolute values and prefix to this sum their common sign.*
2. *To add two signed numbers having* ***unlike*** *signs, find the* ***difference*** *between their absolute values and prefix to it the sign of the number which has the greater absolute value.*

These rules can easily be extended to cases of three or more signed numbers. The sum of two or more signed numbers is often called their *algebraic sum.*

2. *Subtraction.* By remembering the relation "subtrahend plus difference equals minuend" we can readily establish the correctness of the following comparative statements:

(1) $(+8)+(-5)=+3.$	(1) $(+8)-(+5)=+3.$
(2) $(+8)+(+5)=+13.$	(2) $(+8)-(-5)=+13.$

In general, $\qquad -(+n)=+(-n),$

and $\qquad -(-n)=+(+n).$

Hence we have the rule:

To subtract one signed number from another, we may change the sign of the subtrahend (mentally) and then proceed as in addition.

Thus,
$$\begin{array}{rr} & +7 \\ \text{minus} & -2 \\ \hline & +9 \end{array} \quad \text{is the same as} \quad \begin{array}{rr} & +7 \\ \text{plus} & +2 \\ \hline & +9 \end{array}$$

3. *Multiplication.* Since positive and negative numbers are opposites, we can readily find illustrations which support the correctness of the following statements:

$$(+5) \times (+3) = +15. \qquad (-5) \times (+3) = -15.$$
$$(+5) \times (-3) = -15. \qquad (-5) \times (-3) = +15.$$

For example, if your expenses are \$8 a day, in three days your expenses will amount to \$24.

That is, $(-8) \times (+3) = -24.$

Again, if your expenses are \$8 a day, three days ago you were \$24 better off than you are today.

That is, $(-8) \times (-3) = +24.$

Hence we have the following rule:

To multiply one signed number by another, find the product of their absolute values, making the result ***positive*** *if the two signed numbers have* ***like*** *signs, and* ***negative*** *if they have* ***unlike*** *signs.*

This rule can readily be extended to the case of three or more signed numbers. It will be seen that if the number of negative factors is *odd*, the product is *negative*; if *even*, the product is *positive*.

Thus, $(-2) \times (-3) \times (-5) = (+6) \times (-5) = -30,$

and $(-2) \times (+3) \times (-5) = (-6) \times (-5) = +30.$

4. *Division.* Since division is the inverse of multiplication, it follows that *if two algebraic numbers have* ***like*** *signs, their quotient is* ***positive****, but if they have* ***unlike*** *signs, their quotient is* ***negative.***

Thus, $(-15) \div (-5) = +3$, but $(-15) \div (+5) = -3.$

6. Double Use of the Plus and Minus Signs. The signs + and − may be regarded either as *signs of operation* or as *signs of opposition.* It can be shown that in any given case both interpretations lead to the same result.

Illustrative Example. Find the algebraic sum of $+10$, -5, and -2.

Solution. The required sum may be expressed as follows:

(1) $(+10) + (-5) + (-2)$.

But this statement may evidently also be written as:

(2) $(+10) - (+5) - (+2)$.

Since $+10$, $+5$, and $+2$ have the same meaning as 10, 5, and 2, respectively, we see that statement (2) may be written in the form

(3) $10 - 5 - 2$,

which equals 3.

Hence the signs which in the original example were *signs of opposition* have become *signs of operation.* Similarly, by reversing the three steps given above, we can show that the expression $10 - 5 - 2$ may be regarded as the sum of $+10$, -5, and -2. That is, *the given expression may be regarded as the sum of a series of signed numbers.*

Exercises. *Using Signed Numbers*

Find the sum, both differences [for example, $+20 - (+10)$ and $+10 - (+20)$], the product, and the quotient (upper number divided by lower number) in each of the following cases:

1.	2.	3.	4.	5.
$+20$	-16	$+40$	-4	0
$+10$	-2	-5	-1	16

Find the missing number in each case:

6. $+8+(\quad)=-11$.
7. $-12-(\quad)=16$.
8. $-2\times(\quad)=-8$.
9. $-14\div(\quad)=-7$.

Find the product in each case:

10. $(-6)(+2)(-1)(+3)$.
11. $(+20)(-4)(-2)(-3)$.
12. $(-3)(-3)(-3)(-3)$.
13. $(-1)(-1)(-1)(-1)$.

If $x = 3, y = -2$, *and* $z = 1$, *evaluate* * *each of the following:*

14. $4x + 7y - 8z$.
15. $x^2 + y^2 + z^2$.
16. $10xy - y + 9yz$.
17. $x^3 + y^3 + z^3$.
18. $x^2 - xy + y^2$.
19. $3(x + y) - 6(y + z)$.

20. If $a = 6$ and $b = -5$, which expression is greater, and by how much: (1) $a^2 - 4ab - 3$; (2) $b^2 - 5ab + 4$?

3. Fundamental Laws

7. Definitions. An algebraic expression whose parts are not separated by the sign + or − is called a *monomial* or a *term.*

There are *simple* terms such as $5h$, x^2, $4r^3$, $2ab$, and $\frac{n}{3}$, and *compound* terms such as $2(w + l)$ and $\frac{b + b'}{2}$.

When a term is the product of two or more numbers, each number is called a *factor* of that product.

Monomials which have the same literal factors are called *similar terms* or *like terms.*

Thus, $7xy$ and $-5xy$ are similar terms.

Any factor in a term is called the *coefficient* of the product of the remaining factors. However, the word "coefficient" is ordinarily used in the sense of *numerical coefficient.* In any term whose numerical coefficient is not expressed, the coefficient 1 is understood.

Thus, in the term $4ac$, we usually say that 4 is the coefficient of ac, although it is just as correct to say that a is the coefficient of $4c$.

* *Evaluation* is the process of putting a number in place of a letter in a given algebraic expression, and thus finding the corresponding numerical value of the expression.

Thus, to evaluate $A = 2a^2 + 6ab - 5b^2$ when $a = 4$ and $b = 2$, we proceed as follows:

$$A = 2 \times 4^2 + 6 \times 4 \times 2 - 5 \times 2^2 = 2 \times 16 + 48 - 5 \times 4$$
$$= 32 + 48 - 20 = 60.$$

An expression consisting of two or more terms is called a *polynomial.*

A polynomial containing two terms is called a *binomial,* and one containing three terms is called a *trinomial.*

8. Laws of Order (Commutative Laws). Let it be required to find the *sum* of 4, 5, and 6.

Evidently, $4+5+6=6+5+4=6+4+5=4+6+5$, and so on. The terms may be added in any order.

Likewise,

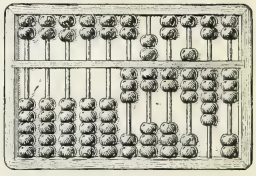
$3a+4a+2a=3a+2a+4a$.

In general,

$$a+b=b+a.$$

That is, *in addition the addends may be combined in any order.*

This is called the *Law of Order for Addition.*



Modern Chinese abacus

The *suan-pan* was used in China as early as the twelfth century for ordinary computations such as those suggested on this page, and it is still in use there

Again, in finding the *product* of 2, 7, and 5, we may write $2\times7\times5=5\times7\times2$, and so on. In general,

$$a\cdot b=b\cdot a.$$

That is, *in multiplication the factors may be combined in any order.*

This is called the *Law of Order for Multiplication.*

9. Laws of Grouping (Associative Laws). Observe that in finding the *sum* of 5, 4, and 6, we may write $5+4+6=(5+4)+6$ or $5+(4+6)$. In general,

$$a+b+c=(a+b)+c=a+(b+c).$$

That is, *in addition the addends may be grouped in any manner.*

This is called the *Law of Grouping for Addition.*

Again, in finding the *product* of 3, 4, and 5, we may write $3 \times 4 \times 5 = (3 \times 4) \times 5$ or $3 \times (4 \times 5)$. In general,

$$abc = (ab)c = a(bc).$$

Thus, *in multiplication the factors may be grouped in any manner.* This is called the *Law of Grouping for Multiplication.*

NOTE. In referring to the laws of order and grouping, it will be found convenient to use the abbreviations O. and G. respectively.

10. Laws of Distribution. Observe that in multiplying 324 by 2, the product is really found by multiplying 4, 20, and 300 successively by 2 and adding the partial products.

That is, $2(300 + 20 + 4) = 600 + 40 + 8 = 648.$

Likewise, $2\,a(b + c + 5) = 2\,ab + 2\,ac + 10\,a.$

In general, $n(a + b - c) = na + nb - nc.$

That is, *to multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, uniting the resulting terms by their proper signs.*

This is called the *Law of Distribution for Multiplication.*

Again, the *quotient* of $96 \div 3$ is found by dividing 90 and 6, successively, by 3 and adding the partial quotients.

That is, $\dfrac{90 + 6}{3} = \dfrac{90}{3} + \dfrac{6}{3} = 30 + 2 = 32.$

Likewise, $\dfrac{3\,a + 6\,b}{3} = \dfrac{3\,a}{3} + \dfrac{6\,b}{3} = a + 2\,b.$

In general,

$$\frac{na + nb - nc}{n} = \frac{na}{n} + \frac{nb}{n} - \frac{nc}{n} = a + b - c.$$

That is, *to divide a polynomial by a monomial, divide each term of the polynomial by the monomial, uniting the resulting terms by their proper signs.*

This is called the *Law of Distribution for Division.*

Exercises. *Applying the Fundamental Laws*

Combine the similar terms in each of the following cases:

1. $3a + 4 + 2a - 3 - a$.
2. $10l + 5w + 10l + 5w$.
3. $x + 11y - 17z + 21x - 14y + 7z$.
4. $5a + 7a + 3b - 3a - b + 2a$.
5. $6l + 9w - 10h - 9w + 15h + 12l$.
6. $-xy - 7xy + z - 3z + 4xy + 6z$.

Using the Rules of Order and Grouping, find the following products:

7. $37 \times 2 \times 5$.
8. $25a \times 17 \times 4b$.
9. $6 \times 12\frac{1}{2} \times 8 \times 16\frac{2}{3}$.
10. $33\frac{1}{3} \times 20 \times 3 \times 5$.
11. $1.75 \times 6 \times 4 \times 0.15$.
12. $40l \times 37w \times 25h$.

Perform the indicated multiplications:

13. $7(x + y - z)$.
14. $11(4a - 3b + 10c)$.
15. $-15(-20n + 9)$.
16. $p(1 + rt)$.
17. $2\pi(R + r)$.
18. $\frac{1}{2}h(b + b')$.

Perform the indicated divisions:

19. $\dfrac{15w - 10}{5}$.
20. $\dfrac{4s + 2\pi r}{2}$.
21. $\dfrac{ab + ac + a}{a}$.
22. $\dfrac{6nr - 9n - 15nt}{3n}$.
23. $\dfrac{24ax - 16bx + 8cx}{-8x}$.
24. $\dfrac{-35lw + 7w + 21wt}{7w}$.

25. The formula for simple interest is $i = prt$. Write the expression prt in five other ways.

11. Laws of Exponents. Since $a^3 = a \cdot a \cdot a$, and $a^2 = a \cdot a$, we have

$$a^3 \times a^2 = (a \cdot a \cdot a) \times (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a = a^5.$$

That is, $a^3 \times a^2 = a^{3+2} = a^5$.

In general, $a^m \times a^n = a^{m+n}$.

That is, *the exponent of the product of powers having like bases is the sum of the original exponents.*

This is called the *Law of Exponents for Multiplication.*

On the other hand, when dividing a^5 by a^2, we have

$$\frac{a^5}{a^2} = \frac{\cancel{a} \cdot \cancel{a} \cdot a \cdot a \cdot a}{\cancel{a} \cdot \cancel{a}} = a^3.$$

In brief form, $\frac{a^5}{a^2} = a^{5-2} = a^3$.

In general, $\frac{a^m}{a^n} = a^{m-n}$.

That is, *the exponent of the quotient of powers having like bases is found by subtracting the exponent of the divisor from that of the dividend.*

This is called the *Law of Exponents for Division.*

NOTE. The quotient of powers having like bases and exponents is 1. A second method of dealing with such quotients will be explained in Chapter VI.

Exercises. *Applying the Rules of Exponents*

Perform the indicated operations in the following cases:

1. $a^5 \cdot a^4$.	5. $a^5 \div a^2$.	9. $(-n^3) \cdot (-n^8)$.
2. $x^2 \cdot (-x^6)$.	6. $2^3 \cdot 2^4$.	10. $r^2 \cdot r \cdot r^4$.
3. $x^n \cdot x$.	7. $2^5 \div 2^3$.	11. $x^{2a} \cdot x^{3a}$.
4. $x^a \cdot x^b$.	8. $x^4 \div x^4$.	12. $x^{2a} \cdot (-x^a)$.

13. $(-a^{2m}) \cdot a^{3n}$.

14. $x^8 \div (-x^7)$.

15. $x^{n+5} \div x^3$.

16. $x^{5a} \div x^{3a}$.

17. $\dfrac{(x^3)^2(xy^3)}{x^2y}$.

18. $(-n^5) \div (-n)$.

19. $(-a^{3x+2}) \div (a^{3x})$.

20. $3^5 \div 3^4$.

21. $2^7 \div 2^5$.

22. $n^5 \cdot n^7 \div n^8$.

23. $a^6 \div a^4 \cdot a^5$.

24. $(-a^2b^3)(ab^2)$.

25. $(2^3\, a^4b^2)(2\, b^5c)$.

26. $\dfrac{2^2\, x^5y^3 \cdot 2^4\, x^2y^2}{-2^3\, x^4y}$.

4. Addition and Subtraction

12. Addition of Monomials and of Polynomials. 1. Observe the following illustrations of the addition of similar terms:

$4\,a$	$7\,w^2$	$-10\,ab$	$-9.1\,x^3$	$\frac{1}{2}\,ab^2$
$5\,a$	$-3\,w^2$	$16\,ab$	$-7.2\,x^3$	$\frac{1}{3}\,ab^2$
$9\,a$	$4\,w^2$	$6\,ab$	$-16.3\,x^3$	$\frac{5}{6}\,ab^2$

That is, *to add similar terms, find the algebraic sum of the coefficients and multiply it by the common literal factor.*

Note. This statement will be called the *Rule of Similar Terms*, and the abbreviation S.T. may be used as a short way of referring to it. Applications of this rule occurred in the group of exercises on page 24.

2. When the terms to be combined are *dissimilar*, their sum can only be *indicated* by means of *plus* signs.

Illustrative Example. Add $4\,ab^2$, $-3\,ab^2$, bc.

Solution. $4\,ab^2 + (-3\,ab^2) + bc = ab^2 + bc$.

3. When two or more *polynomials* are to be added, the terms should be so arranged that similar terms may be written in the same column.

Illustrative Example. Find the sum of $3x - 5y + z$, $6z - 2x - 4y$, and $y + 5x + 3z$, and check the result.

Solution. The following *vertical* arrangement will be found convenient:

Operation	*Check.* Let $x = 2$, $y = 3$, and $z = 4$.
$3x - 5y + z$	$6 - 15 + 4 = -5$
$-2x - 4y + 6z$	$-4 - 12 + 24 = +8$
$5x + y + 3z$	$10 + 3 + 12 = +25$
$6x - 8y + 10z$	$12 - 24 + 40 = +28$

NOTE. In checking, the number 1 should not be substituted for any letter having an exponent, since any power of 1 is always equal to 1. Hence, in the case of an incorrect exponent the substitution of 1 will not detect the fact that a mistake has been made.

Exercises. *Addition of Monomials and Polynomials*

Add in each of the following cases:

1. $12x$ / $5x$

2. $-9xy$ / $5xy$

3. $19x^a$ / $-x^a$

4. $24a^3b$ / $-16a^3b$ / $-5a^3b$

5. $-4.5ax - 17$ / $3.5ax - 2.5$

6. $8a^2 + 5ab - 13b^2$ / $13a^2 - 5ab + 6b^2$

Find the sum in each of the following cases:

7. $3lw + lw$.

8. $9s^2 + (-16s^2)$.

9. $-4bh + (-12bh) + (-4bh)$.

10. $23.2rt + (-7.8rt) + (-6.5rt)$.

Find the sum of the polynomials given in each of the following cases, and check each result:

11. $12c + 5d$, $10c - 3d$, $-15c - 6d$.

12. $a^2 - 3a + 1$, $-2a^2 - 7a + 6$, $3a^2 + 5a - 4$.

13. **Subtraction of Monomials and of Polynomials.** In finding the difference between two similar terms, we make use of the fact (see page 18) that $-(+n) = +(-n)$, and that $-(-n) = +(+n)$.

That is, *to subtract a term from one which is similar to it, mentally change the sign of the subtrahend and proceed as in addition.*

A similar plan is followed in the subtraction of one *polynomial* from another.

Illustrative Example. Subtract $3x^2 + 7 - 5x$ from $2x^2 + 3x - 4$.

Solution. The following arrangement will be found convenient:

Operation	*Check.* Let $x = 3$.
$2x^2 + 3x - 4$	$18 + 9 - 4 = +23$
$3x^2 - 5x + 7$	$27 - 15 + 7 = +19$
$-x^2 + 8x - 11$	$-9 + 24 - 11 = +4$

Checking. The result may be checked either as in arithmetic, or by numerical substitution as shown above.

Exercises. *Subtraction of Monomials and Polynomials*

Subtract in each of the following cases:

1. $29b$ from $33b$
2. a^2x from $7a^2x$
3. $-3.4y$ from $1.4y$
4. $9w$ from $-15w$
5. $-17x^n$ from $-26x^n$
6. $-43t$ from 0
7. 0 from $-5x^3$
8. $9a^{n+1}$ from $-a^{n+1}$
9. $12c + 16d$ from $6c - 7d$
10. $3a^3b - 13b^2$ from $3a^3b + 6b^2$
11. $8r - 11s + 4t$ from $-10r + 11s - 4t$
12. $19x^3 - 5x$ from $-x^3 + 4x^2 + 3x$

5. Multiplication and Division

14. Multiplication by a Monomial. The multiplication of monomials involves the laws of signs, order, grouping, and exponents, which have already been considered.

Thus, to multiply $(+5\,ab^2)$ by $(-3\,a^3b^4c)$, we have

$$(+5)\cdot a\cdot b^2\cdot(-3)\cdot a^3\cdot b^4\cdot c = (+5)\cdot(-3)\cdot a\cdot a^3\cdot b^2\cdot b^4\cdot c$$

[Law of Order]

$$= [(+5)\cdot(-3)]\cdot[a\cdot a^3]\cdot[b^2\cdot b^4]\cdot c$$

[Law of Grouping]

$$= -15\,a^4b^6c.$$

[Laws of Signs and of Exponents]

To multiply a *polynomial* by a *monomial*, the Law of Distribution must be used, as we have already seen. The work may be arranged either *horizontally* or *vertically*, as shown below.

Horizontal arrangement:

$$-4\,x(x^2-3\,x+1) = -4\,x^3+12\,x^2-4\,x.$$

Vertical arrangement:

$$\begin{array}{r} x^2-3\,x+1 \\ \underline{\qquad\qquad -4\,x} \\ -4\,x^3+12\,x^2-4\,x \end{array}$$

Exercises. *Multiplication by Monomials*

Perform the indicated multiplications:

1. $3\,a^2\cdot 5\,a^3$.
2. $5\,x\cdot x^3$.
3. $(-t)(t^4)$.
4. $(3\,x^2)^2$.
5. $(4\,x^n)^2$.
6. $(-a^{2x})^3$.
7. $(a^{n+1})^2$.
8. $(1.2\,a^2b^2)(-4\,bc)$.
9. $(2\,a^{3n})(3\,a^{2n})$.
10. $(-5\,x^n)(5\,x^n)$.
11. $(a^{2n+1})(a^{3n+2})$.
12. $(4\,x^{a-1})(-6\,x^{a+1})$.
13. $(7\,x^n)(5\,x^2)$.
14. $(-11\,xy^2)(8\,x^ay^b)$.
15. $(-5\,w)(-5\,w)$.
16. $(5\,x)(6\,x^3)^2$.
17. $(2\,a^2b)(3\,ab^2)$.
18. $(9\,lw)(5\,l^3w^2)$.
19. $(-7s^2)(-0.8\,t^2)$.
20. $(x^{2a}y^{b+2})^3$.
21. $(5\,x^n)(2\,xy)^3$.
22. $5\,x(6\,x^2-4\,x+3)$.
23. $2.5\,x^n(-6\,x^n+5\,x^{2n}y)$.
24. $-3\,a^2(5\,a^2+7\,a-8)$.
25. $7\,x^2y^3(4\,x^2-12\,xy+9\,y^2)$.

15. Removing Parentheses. When multiplying a polynomial by a monomial horizontally, the application of the Law of Distribution makes possible the removal of the parentheses inclosing the given polynomial. The following points, however, should be kept in mind:

1. *Be very sure to apply correctly the Rule of Signs for Multiplication.*

Thus, $4(x + y - z) = 4x + 4y - 4z.$

Similarly, $-3x(x^2 - 4x + 2) = -3x^3 + 12x^2 - 6x.$

NOTE. You will observe that when the parentheses which are removed are preceded by a *plus* sign, the signs of the terms inclosed within the parentheses remain *unchanged*; but when they are preceded by a *minus* sign, the signs of the terms inclosed must be *changed*, from + to − and from − to +.

2. *Remember that any parentheses without an indicated coefficient have in reality a coefficient of 1.*

Thus, $(x + 4y) = 1(x + 4y) = x + 4y.$

Also, $-(a - 3b) = -1(a - 3b) = -a + 3b.$

3. *Keep in mind the proper order of operations.*

Thus, $10 - 3(c + 3d) = 10 - 3c - 9d.$

Hence $10 - 3(c + 3d)$ *does not equal* $7(c + 3d).$

4. *In the case of parentheses inclosed within other parentheses, remove one set at a time, starting with the innermost.*

Thus, $-4x[x - (3 - y)] = -4x[x - 3 + y] = -4x^2 + 12x - 4xy.$

Exercises. *Removing Parentheses*

Remove the parentheses and combine similar terms:

1. $5(2a + 3b) - 2(4a - 5b).$
2. $3a(6a^2 - 5a + 7) - 8a^2(4a + 9) + 17a^2.$
3. $-2x(11a - 12) + (5ax - 3x).$
4. $1.4n + (2n - 1.7) - 0.3(9n + 4).$

5. $40\,l - [(16\,l - 20\,w) + (12\,w - 9\,l)]$.

6. $(18\,x + 11\,y) + [13\,y - 5(14\,x + 10\,y) - 9\,y]$.

7. $[(31\,a - 36\,b) - 21\,c] - [19\,a - (39\,c + 43\,b)]$.

8. $\left(\frac{k}{4} + \frac{l}{3} - \frac{m}{2}\right) + (3\frac{1}{4}\,k + 3\frac{2}{3}\,l) - 3(1\frac{3}{4}\,k - 5\frac{1}{2}\,m)$.

9. $-20\,p + \{54\,p - 2[37\,q - (28\,r + 12\,p)]\}$.

10. $2\,a - \{7\,b + [c - (5\,a - b)] - [(4\,b - c) + 6\,a] + 3\,c\}$.

16. Multiplication of a Polynomial by a Polynomial. The product of two polynomials is found by a repeated application of the Law of Distribution.

For example,

$$\begin{aligned}(3x+5)(x^2-2x+4) &= 3\,x(x^2 - 2\,x + 4) + 5(x^2 - 2\,x + 4)\\ &= 3\,x^3 - 6\,x^2 + 12\,x + 5\,x^2 - 10\,x + 20\\ &= 3\,x^3 - x^2 + 2\,x + 20.\end{aligned}$$

That is, *each term of the first polynomial must be multiplied by each term of the second, and the partial products must then be combined.*

The work is usually arranged *vertically*, as follows:

Operation	*Check.* Let $x = 2$.
$x^2 - 2\,x + 4$	$4 - 4 + 4 = 4$
$3\,x + 5$	$6 + 5 = 11$
$3\,x^3 - 6\,x^2 + 12\,x$	44
$+ 5\,x^2 - 10\,x + 20$	
$3\,x^3 - x^2 + 2\,x + 20$	$24 - 4 + 4 + 20 = 44$

The result may be checked by numerical substitution, as shown above.

In the multiplication of polynomials it is usually advisable to arrange the terms of both expressions according to the *descending* or the *ascending* powers of the same letter. Observe that this plan is followed in the preceding example

Exercises. *Multiplying a Polynomial by a Polynomial*

Multiply:

1. $4a + 7$ by $3a - 5$.
2. $8x - 15$ by $5x + 6$.
3. $16 + 4n$ by $4 - n$.
4. $9x - 12y$ by $8x - y$.
5. $x^{2a} - 3x^a - 4$ by $x^a - 6$.
6. $2a^3 - 3b^2$ by $2a^3 + 3b^2$.
7. $6a^{n+2} - 9a^{n+1}$ by $5a^{n+1} + 3a^n$.
8. $1.2x^{2n} - 4.5x^n + 0.2$ by $7x^n - 0.8$.
9. $\frac{2}{3}r - \frac{1}{2}s + \frac{1}{4}t$ by $6r - 12s$.
10. $7.5a^2b + 0.25ab^2$ by $-0.32ab^2 + 4.4b^3$.

Multiply in each case and check the result:

11. $(x^2 - 5x + 6)(2x - 3)$.
12. $(3a^2 + 7a - 4)(a^2 - 9)$.
13. $(x - 2y + 3z)^2$.
14. $(x^2 - 5 + 3x^3 - 2x)(x^2 - 6 - 5x)$.

SUGGESTION. Arrange the terms of both polynomials in the same way according to the powers of x.

15. $(5n^2 - 6n + 8)(3n + n^2 - 4)$.
16. $(\frac{3}{4} + \frac{1}{4}n^2 - \frac{1}{2}n)(\frac{3}{4}n - \frac{1}{2}n^2 + \frac{1}{4})$.
17. $(-4s^2 - 9r^2)(3r - 2s)(2s + 3r)$.

If $A = 2x^2 + 5x - 3$, $B = 3x^2 - 4x + 2$, *and* $C = 5x^2 - x + 1$, *find*

18. $2A + 3B$.
19. $3B - 4C$.
20. AB.
21. AC.
22. $4AB - 5AC$
23. $BC - 3AB$.

17. Division by a Monomial. The division of an algebraic expression by a monomial requires the use of one or more of the laws of division which have been previously stated; namely, the laws of signs, exponents, and distribution.

Thus, $$\frac{-15\,b^3h^5}{+5\,bh^2} = -3\,b^2h^3,$$

and $$\frac{3\,x^3 - 6\,x^2 - 12\,x}{-3\,x} = -x^2 + 2\,x + 4.$$

That is, *to divide any algebraic expression by a monomial, divide each term of the dividend by the divisor, uniting the resulting terms by their proper signs.*

Exercises. *Division by a Monomial*

Numbers 1 to 7, oral

Divide:

1. $24\,a^3$ by $6\,a^2$.
2. $-15\,xy^3z^2$ by $-15\,xyz^2$.
3. $4.2\,r^2s^2$ by $14\,s$.
4. $4.5\,a^5b^3c^4$ by $0.06\,a^3c^4$.
5. $28\,a^{2n}$ by $7\,a^n$.
6. $48\,a^{3x+1}$ by $8\,a^{2x+1}$.
7. $-46\,x^{3n+2}$ by $23\,x^{n-1}$.
8. $2^4 \cdot 3^5\,p^2qr^4$ by $2 \cdot 3^4\,pqr^2$.

Perform the indicated division in each case:

9. $\dfrac{54\,a^2b^3}{18\,ab}$.
10. $\dfrac{-39\,mn^4}{-13\,m}$.
11. $\dfrac{-56\,x^3y}{7\,x^2y}$.
12. $\dfrac{-27\,a^{5n}}{9\,a^{2n}}$.
13. $\dfrac{34\,x^ay^{2b}}{17\,x^ay^b}$.
14. $\dfrac{2^6\,a^{3n+1}}{2^3\,a^2}$.
15. $(32\,x^3y + 8\,x^2) \div 8\,x^2$.
16. $(\frac{2}{3}\,r^2s^2 - \frac{4}{5}\,rs^3) \div 2\,rs^2$.
17. $\dfrac{19\,l^2w^2h^2 - 13\,lwh}{-lwh}$.
18. $(9\,ab^3 - 12\,a^3b^5) \div 3\,ab^2$.
19. $(\frac{9}{16}\,abc + \frac{3}{4}\,a^2bc) \div 3\,abc$.
20. $\dfrac{60\,c^4d^3 - 36\,c^3d^2 - 12\,cd}{12\,cd}$

18. Division of a Polynomial by a Polynomial. The process of dividing one polynomial by another is very similar to that of long division in arithmetic, as a comparison of the following examples will show:

ARITHMETIC	ALGEBRA
$\begin{array}{r} 213 \\ 32\overline{)6816} \\ \underline{64} \\ 41 \\ \underline{32} \\ 96 \\ \underline{96} \end{array}$	$\begin{array}{r} x^2 - 2x + 4 \\ 3x+2\overline{)3x^3 - 4x^2 + 8x + 8} \\ \underline{3x^3 + 2x^2} \\ -6x^2 + 8x \\ \underline{-6x^2 - 4x} \\ 12x + 8 \\ \underline{12x + 8} \end{array}$
Check.	***Check.*** Let $x = 2$.
[Quotient] 213	$\dfrac{\text{Dividend}}{\text{Divisor}} = \dfrac{24 - 16 + 16 + 8}{6 + 2} = \dfrac{32}{8} = 4.$
[Divisor] 32	Quotient $= 4 - 4 + 4 = 4.$
426	
639	
[Dividend] 6816	

The algebraic result may be checked, as in arithmetic, by using the relation

$$Divisor \times quotient = dividend,$$

or by numerical substitution, as shown above. In the case of a remainder, it is necessary to use the relation

$$\frac{Dividend}{Divisor} = partial\ quotient + \frac{remainder}{divisor}.$$

NOTE. In checking by numerical substitution, care must be exercised not to substitute a value which will cause the divisor to become zero, since division by zero is impossible.

Observe, for example, that in dividing 5 by 0, the indicated quotient $\frac{5}{0}$ would have to be a number which when multiplied by the divisor 0 would equal the dividend 5. No such number can be found. That is, any expression of the form $\frac{n}{0}$ must be excluded from our consideration.

The method used in dividing one polynomial by another may be summarized as follows:

1. *Arrange the terms of the dividend and the divisor according to the descending (or ascending) powers of the same letter.*
2. *Divide the first term of the dividend by the first term of the divisor, thus obtaining the first term of the quotient.*
3. *Multiply the divisor by the term of the quotient just found and subtract the product from the dividend, thus obtaining a new dividend.*
4. *Repeat steps 2 and 3 until there is no remainder, or until in the remainder the exponent of the letter upon which the arrangement is based is less than the largest exponent it has in the divisor.*

Exercises. ***Dividing a Polynomial by a Polynomial***

Divide:

1. $4\,n^2 + 20\,n + 21$ by $2\,n + 7$.
2. $6\,a^2 - 13\,a + 6$ by $2\,a - 3$.
3. $14\,r^2 + 10 - 39\,r$ by $7\,r - 2$.
4. $c^3 - 7\,c^2 + 7\,c + 6$ by $c - 2$.
5. $x^2 + 12\,x^4 - 20$ by $4\,x^2 - 5$.
6. $49\,s + 42 - 35\,s^2$ by $14 - 7\,s$.
7. $21\,t^4 - 30 - 61\,t^2$ by $3\,t^2 - 10$.
8. $2\,x^{2a+2} - 17\,x^{a+1} + 21$ by $x^{a+1} - 7$.
9. $n^3 + n - 10$ by $n - 2$.

SUGGESTION. Write the dividend $n^3 + 0\,n^2 + n - 10$.

Divide in each case, checking each result:

10. $30\,m^2 + 15\,n^2 - 81\,mn$ by $15\,n - 6\,m$.

11. $23\,pq - 15\,p^2 - 4\,q^2$ by $5\,p - q$.

12. $ac - 2\,ad + 3\,bc - 6\,bd$ by $a + 3\,b$.

13. $49\,x^4 + 34\,x^2y^2 + 25\,y^4$ by $7\,x^2 - 6\,xy + 5\,y^2$.

14. $4\,a^2 - 12\,a + 9 - 16\,c^2$ by $2\,a - 3 + 4\,c$.

15. $21\,x^3 + 10\,x^2 - 31\,x + 13$ by $3\,x - 2$.

Solution.

$$
\begin{array}{r|l}
3\,x - 2 & 7\,x^2 + 8\,x - 5 \\
\hline
 & 21\,x^3 + 10\,x^2 - 31\,x + 13 \\
 & \underline{21\,x^3 - 14\,x^2} \\
 & \quad 24\,x^2 - 31\,x \\
 & \quad \underline{24\,x^2 - 16\,x} \\
 & \quad\quad -15\,x + 13 \\
 & \quad\quad \underline{-15\,x + 10} \\
 & \quad\quad\quad +\ 3 \quad \text{[Remainder]}
\end{array}
$$

Hence the complete quotient is $7\,x^2 + 8\,x - 5 + \dfrac{3}{3\,x - 2}$.

Check. Let $x = 2$. $\dfrac{\text{Dividend}}{\text{Divisor}} = \dfrac{168 + 40 - 62 + 13}{6 - 2} = \dfrac{159}{4} = 39\frac{3}{4}$.

$$\text{Quotient} = 28 + 16 - 5 + \frac{3}{6 - 2} = 39 + \tfrac{3}{4} = 39\tfrac{3}{4}.$$

Divide and check:

16. $12\,n^3 - 19\,n^2 + 13\,n - 11$ by $3\,n - 4$.

17. $14\,r^3 - 23\,r^2 - 38\,r + 17$ by $2\,r - 5$.

18. $10\,x^3 - x^2y - 4\,xy^2 + 2\,y^3$ by $2\,x - y$.

19. $a^4 - 2\,a^2b^2 + 8\,ab^3 - 3\,b^4$ by $a^2 + 2\,ab - b^2$.

20. $6\,c^4 - 83\,c^2d^2 - 3\,cd^3 + 35\,d^4$ by $2\,c^2 - 6\,cd - 5\,d^2$.

21. $4\,m^4 - 8\,m^2 + 2\,m + 49$ by $2\,m^2 - 6\,m + 7$.

22. $81\,x^4 + 11\,x^2y^2 + 4\,y^4$ by $2\,y^2 + 9\,x^2 + 5\,xy$.

6. Solving Simple Equations

19. Solving Equations by the Rules of Equality. It will be well to recall the following definitions:

An *equality* is a statement showing that two expressions stand for the same number or value. Thus, $3 + 2 = 5$ is an equality.

An *equation* is an equality that asks a question. Thus, $3n - 5 = 16$ is a short way of writing the question "Three times what number, when 5 is subtracted, will give the total result 16?"

An *equation of the first degree in one unknown* is one of such form that (1) the unknown appears only with the exponent 1; (2) the unknown does not appear in any denominator when the equation is written in its simplest form. Such equations are commonly called *simple equations*. For example, $5n + 2 = 12$, $7x - 3 = 4x + 17$, and $21 = 5 - \frac{4y}{3}$ are equations of the first degree.

A *formula* is an equality which states a generalized relation or rule in the shorthand of algebra. Thus, $A = \pi r^2$ is a formula.

Simple equations are solved by the rules of equality. These rules may be summarized as follows:

*If equals are increased, or decreased, or multiplied, or divided by equals, the results are equal.**

To make sure that the correct value of the missing number has been found, the result obtained by solving an equation should be substituted for the missing number in each side of the original equation. If the value of each side is found to be the same, we know that the equation balances. The result is said to *check*, and may be called a *root*.

Illustrative Example 1. Solve the equation $n - 6 = 8$.

Solution. To find the value of n, 6 must be *added* to each member.

(1)	$n - 6 = 8.$	Given equation.
(2)	$n = 14.$	Adding 6 to both members.

Check. Left side (L.S.) $= 14 - 6 = 8$; Right side (R.S.) $= 8$.
Hence $n = 14$ satisfies the equation.

* Division by zero is excluded for the reason explained on page 34.

Illustrative Example 2. Solve the equation $a + 9 = 5$.

Solution. To find the value of a, 9 must be *subtracted* from each member of the equation.

(1) $a + 9 = 5$.	Given equation.
(2) $a = -4$.	S_9.

Check. L.S. $= -4 + 9 = 5$; R.S. $= 5$.
Hence $a = -4$ satisfies the equation.

NOTE. The abbreviation S_9 means that we have subtracted 9 from each member of the equation. A similar method of notation will be used in the case of the other operations.

Illustrative Example 3. Solve the equation $\frac{x}{3} = 6$.

Solution. To find the value of x, each member of the equation must be *multiplied* by 3.

(1) $\frac{x}{3} = 6$.	Given equation.
(2) $x = 18$.	M_3.

Check. L.S. $= \frac{18}{3} = 6$; R.S. $= 6$.
Hence $x = 18$ satisfies the equation.

Illustrative Example 4. Solve the equation $4y = 14$.

Solution. To find the value of y, each member of the equation must be *divided* by 4.

(1) $4y = 14$.	Given equation.
(2) $y = 3\frac{1}{2}$.	D_4.

Check. L.S. $= 4 \times 3\frac{1}{2} = 14$; R.S. $= 14$.
Hence $y = 3\frac{1}{2}$ satisfies the equation.

Illustrative Example 5. Solve the equation $5x - 3 = 2x + 9$.

Solution.

(1) $5x - 3 = 2x + 9$.	Given equation.
(2) $3x - 3 = 9$.	S_{2x}.
(3) $3x = 12$.	A_3.
(4) $x = 4$.	D_3.

Check. L.S. $= 20 - 3 = 17$; R.S. $= 8 + 9 = 17$.
Hence $x = 4$ satisfies the equation.

Exercises. *Applying the Rules of Equality*

Numbers 1 to 12, oral

Solve the following equations, and check each result:

1. $5n = 30$.	7. $\frac{1}{4} + a = 6\frac{3}{4}$.	13. $2\frac{1}{2}\,m = 15$.
2. $28 = 4x$.	8. $d + 11 = 11$.	14. $0.25\,v = 5$.
3. $8a = 12$.	9. $h - 3 = 7$.	15. $3s = s + 4$.
4. $3r + 4r = 21$.	10. $r - 2.1 = 8$.	16. $9k - 14 = 2k$.
5. $7n - 5n = 8$.	11. $12 = x - 8$.	17. $4t - 7 = 2t + 3$.
6. $\frac{1}{2}a = 5$.	12. $2a + 5 = 11$.	18. $3x - 5 = 7 - x$.

20. Special Methods of Solving Equations. There are certain special methods, based on the rules of equality and on the definition of an equality, which enable us often to solve equations more quickly. You have already learned these methods, but they are re-stated here so that you may review them:

1. *The members of an equation may be interchanged.*

Thus, $12 = 7x + 5$ may be written in the form $7x + 5 = 12$.

2. *A term may be dropped from either member of an equation, provided the corresponding opposite term is written in the other member.*

NOTE. This statement is often called the *Rule of Transposition*, since it looks as if the original term, *with an opposite sign*, had been moved, or *transposed*, from one member of the equation to the other. We shall use the abbreviation Tr. (for "transposition") as a short way of referring to this rule.

Thus, $3x + 2 = 11 - x$ may be written in the form $3x + x = 11 - 2$.

3. *Multiplying or dividing each member of an equation by -1 changes the sign of each term in the given equation.*

Thus, $8 - 6n = -2n - 16$ may be transformed into $-8 + 6n = 2n + 16$.

Illustrative Example. Solve the equation

$$5x + 8 - 4x = 2x - 12.$$

Solution. Using the methods explained above, we have

(1)	$5x + 8 - 4x = 2x - 12.$	Given.
(2)	$5x - 4x - 2x = -12 - 8.$	Tr.
(3)	$-x = -20.$	S.T.
(4)	$x = 20.$	M_{-1} (or D_{-1}).

Check. Substitute 20 for x in each side of the original equation.

HISTORICAL NOTE. There is evidence of algebraic thinking in early Egyptian documents as far back as 1600 B.C. The term "algebra," however, appears to be derived from the title of one of the works of the Arabian mathematician al-Khowârizmi, who lived in the ninth century at the court of the son of Harun al-Rashid, the caliph famous from the Arabian Nights tales. The treatise of al-Khowârizmî was entitled *al-jabr wa'l-muqâbalah.* This phrase has been variously translated by scholars, but it may now be regarded as settled that the word *al-jabr* has a Babylonian origin and really means the SCIENCE OF EQUATIONS, while *al-muqâbalah* is merely the Arabic translation of the term *al-jabr.*

Exercises. *Applying the Rule of Transposition*

Numbers 1 to 14, oral

Solve each of the following equations, and check each result:

1. $5a = 2.1 - 2a.$
2. $8x = 35 + 3x.$
3. $17 = 4x + 5.$
4. $6a - 13 = 11.$
5. $6 = 6 - 11h.$
6. $7 = 8w + 3.$
7. $14t + 11 - 8t = 11.$
8. $v - 3 - 5v = 13.$
9. $8k + 7 = 6k - 5.$
10. $4y - 6 = 9y + 4.$
11. $5n + 8 = 7n - 2.$
12. $32k - 6 = 4k + 1.$
13. $5 - 4y = 1 - 6y.$
14. $15n + 4 = 11 - 6n.$
15. $12n + 17 = 11 - 4n + 18.$
16. $12a - 21 = 12 + 4a + 15.$

17. $2r + 0.7 - 1.2r = 0.45r - 0.35$.

18. $3 - 1.1t = 1.8 - 2.2t - 2.4 + 0.7t$.

19. $9 + 2x - 3 = -15 - 3x - 9$.

20. $10s + 5 - 2s = 6s + 12 - 2s - 5$.

21. $13 - 24x = 12 - 48x + 5 + 40x$.

22. $1.1w + 0.17 = 4.4 - w - 0.03$.

23. $12k + 19 - k - 7 = 16 - 3k - 4 + 30k$.

24. A man bought two books for \$4.25. One of the books cost 75¢ more than the other. Find the cost of each.

25. The sum of three numbers is 11. The second is 9 less than the first, and the third is 1 less than five times the first. What are the numbers?

21. Equations involving Parentheses. When a problem involves more than one unknown quantity, it is often necessary to use parentheses in writing the equation which expresses the relationship stated in the problem. In such a case, the parentheses may be removed by applying the Rule of Distribution for Multiplication.

Illustrative Example. Solve the equation

$$5 + 2(x + 3) - (x - 4) = 18.$$

Solution.

(1)	$5 + 2(x + 3) - (x - 4) = 18.$	Given.
(2)	$5 + 2x + 6 - x + 4 = 18.$	Why?
(3)	$x + 15 = 18.$	S.T.
(4)	$x = 3.$	S_{15}.

Check. When $x = 3$,

L.S. $= 5 + 2(6) - (-1) = 5 + 12 + 1 = 18$; R.S. $= 18$.

Hence $x = 3$ satisfies the equation.

Exercises. *Equations containing Parentheses*

Numbers 1 to 5, oral

Solve each of the following equations, and check each result:

1. $5(a+1)=20$.
2. $3(2n-5)=9$.
3. $4(3n-4)-8=0$.
4. $-7(2K-3)+21=0$.
5. $2x+3(x-2)=4$.
6. $11r-6(3r-2)=12$.
7. $15w-(7w+8)=16$.
8. $9-(21-5y)=8y$.
9. $6(p-2)=3-5(p+14)$.
10. $-4(4a-1)=3-(6+2a)$.
11. $3(2k+1)-(4k-7)=9$.
12. $5(n-1)=12-(17-3n)$.
13. $6-(5-2x)=3+(3x-9)$.
14. $x+(3x-7)=4x-(x+2)$.
15. $4(2x-11)+(3x-22)=0$.
16. $3(5s+1)-(13-5s)=-5s$.
17. $3(3a-1)+4a=6(a+5)-12$.
18. $2n-(5n+32)=22-5(3-2n)$.
19. $5(8y+3)-(4y-2)=12y+(29+8y)$.
20. $(x+1)(x+2)=x^2+17$.
21. $w(3w+5)+7-3(w^2-2w-3)=5$.
22. $(n-4)(n+8)=7-(3-n)(n+5)$.
23. $2r^2-(r+3)(r-5)=10-(5+3r-r^2)$.
24. $3x(2x+7)-[x^2+2(x-4)]=5x^2+13x-16$.

25. A man leaves an estate of $20,000 to his wife, his son, and three daughters. The wife receives $500 more than three times the son's share, and each daughter receives $500 less than the son's share. Find the share of each.

26. A farmer received \$400 for selling his surplus of wheat and oats, 500 bu. in all. For the wheat he received \$1.10 a bushel and for the oats 60¢ a bushel. How many bushels of each did he sell?

22. Equations with Fractional Coefficients. An equation containing fractional coefficients may easily be transformed into one which contains only whole numbers, by multiplying both members of the equation by a number that contains each of the denominators as one of its factors. This process is known as *clearing the equation of fractions.*

Illustrative Example. Solve the equation

$$\frac{3\,n}{4} - 2 = \frac{n}{6} + \frac{1}{3}.$$

Solution. Both members should be multiplied by 12.*

(1)	$\frac{3\,n}{4} - 2 = \frac{n}{6} + \frac{1}{3}.$	Given.
(2)	$12\left(\frac{3\,n}{4} - 2\right) = 12\left(\frac{n}{6} + \frac{1}{3}\right).$	M_{12}.
(3)	$9\,n - 24 = 2\,n + 4.$	Why?
(4)	$9\,n - 2\,n = 4 + 24.$	Tr.
(5)	$7\,n = 28.$	S. T.
(6)	$n = 4.$	D_7.

Check. When $n = 4$,

$$\text{L.S.} = \tfrac{1}{4}^2 - 2 = 3 - 2 = 1;\quad \text{R.S.} = \tfrac{4}{6} + \tfrac{1}{3} = 1.$$

Hence $n = 4$ satisfies the equation.

* Remember that when a fraction is multiplied by a whole number, its numerator only is multiplied by that number. Thus,

$$5 \times \tfrac{2}{3} = \tfrac{10}{3};\quad 7 \times \tfrac{3}{7} = \tfrac{21}{7} = 3;\ \text{and so on.}$$

Similarly, in algebra, $4 \times \frac{x}{3} = \frac{4\,x}{3};\quad 12 \times \frac{3\,x}{4} = \frac{36\,x}{4} = 9\,x;$ and

$$2\left(2 - \frac{2\,x+1}{3}\right) = 2\left[\frac{6-(2\,x+1)}{3}\right] = 2\left[\frac{6-2\,x-1}{3}\right] = 2\left[\frac{5-2\,x}{3}\right] = \frac{10-4\,x}{3}.$$

Exercises. *Equations with Fractional Coefficients*

Numbers 1 to 6, oral

Solve each of the following equations, and check each result:

1. $\frac{1}{2}n = 3$.
2. $\frac{3}{4}w = 6$.
3. $\frac{1}{4}r + 2 = 3$.
4. $\frac{3}{4}y + 8 = 2$.
5. $\frac{2}{3}s - 3 = 1$.
6. $\frac{4}{3}n - 3 = -11$.
7. $12 + \frac{3}{4}x = 0$.
8. $7 - \frac{1}{2}p = 10$.
9. $\frac{1}{2}(x + 8) = 5$.
10. $\frac{5}{3}t + 10 = 0$.
11. $\frac{2x-1}{3} + 7 = 10$.
12. $9x = 8 - \frac{3x-7}{4}$.
13. $\frac{7}{2}k - 35 = 0$.
14. $\frac{1}{2}n - \frac{1}{4}n = 3$.
15. $\frac{3}{4}a - 2 = \frac{1}{2}a + 3$.
16. $\frac{3}{4}(5y - 11) = 18$.
17. $13 - \frac{d+5}{2} = 9$.
18. $\frac{3n+1}{5} - 4n = 7$.
19. $\frac{k}{5} + \frac{k}{3} - \frac{k}{15} = 14$.
20. $\frac{w+10}{6} + 1 - \frac{w}{4} = \frac{4-5w}{6} - 1$.
21. $\frac{2(x-4)}{3} - 1 = \frac{x+5}{3}$.
22. $1.7n - 4.73 - 0.03n = 1.1n - 0.17$.
23. $1.3t + 5.14 = 2.1 - 1.9t - 9.76$.

24. The sum of $\frac{1}{2}$ of a number, $\frac{1}{3}$ of the number, and $\frac{1}{4}$ of the same number is 26. Find the number.

25. A boy receives a sum of money on his twelfth birthday. He spends $\frac{1}{4}$ of it for a small set of tools, $\frac{1}{5}$ of it for a book, and deposits the remainder, amounting to \$5.50, in the bank. How much did he receive?

26. In buying groceries for the home, a housewife pays $\frac{1}{4}$ as much for bread as for fruit, $\frac{7}{8}$ as much for vegetables as for fruit, and $1\frac{1}{2}$ times as much for meat as for fruit. She spends \$2.90 in all. How much does she pay for each item purchased?

23. Literal Equations. An equation in which at least one coefficient or term is represented by a letter, other than the letter representing the unknown quantity, is called a *literal equation.*

For example, $x + 5 = n$, $ax = b$, and $2x - c = x + 3d$ are literal equations in which x represents the unknown quantity.

Such an equation may be solved by applying the same methods as those used in solving *numerical* equations.

NOTE. In literal equations it is customary to represent the unknowns by letters chosen from the *latter* part of the alphabet, while the known quantities are represented by the *first* letters of the alphabet.

Illustrative Example. Solve for y each of the following equations:

$$12y - 21 = 8y + 4 \quad \text{and} \quad 6ay - 7b = 4ay + c.$$

Solution. *(Numerical Equation)*

(1)	$12y - 21 = 8y + 4.$	Given.
(2)	$4y - 21 = 4.$	$S_{8y}.$
(3)	$4y = 25.$	$A_{21}.$
(4)	$y = \frac{25}{4}.$	$D_4.$

Solution. *(Literal Equation)*

(1)	$6ay - 7b = 4ay + c.$	Given.
(2)	$2ay - 7b = c.$	$S_{4ay}.$
(3)	$2ay = c + 7b.$	$A_{7b}.$
(4)	$y = \frac{c + 7b}{2a}.$	$D_{2a}.$

Check. In each case substitute in the original equation the result obtained.

Exercises. *Literal Equations*

Solve the following equations for x or y, and check each result:

1. $nx = 5.$
2. $rx = t.$
3. $x + m = n.$
4. $2y - p = 6.$
5. $ay - b = 0.$
6. $a + by = c.$
7. $m - nx = p.$
8. $3cx + 10c = 4c.$
9. $9r^2 = 6r - 3rx.$
10. $-3ny - 4n^2 = 20n^2 + 5ny.$
11. $5ax - 8a^2b = 6a^2b + 3ax.$

12. $5\,ay - 12 = 2\,ay.$

13. $5 - 2\,bx = 5 + bx.$

14. $4\,nx = 3\,nx - 8\,n.$

15. $3\,ay - ab = 2\,ay - ac.$

16. $\dfrac{y+n}{3} = n.$

17. $\dfrac{3\,x}{4} - \dfrac{x+3\,a}{8} = \dfrac{3\,a}{2}.$

18. $\dfrac{4\,x}{d} = \dfrac{3\,x}{2\,d} - 15.$

19. $\dfrac{y-a}{2\,a} = 3 - \dfrac{y-4\,a}{4\,a}.$

The formulas below may be regarded as literal equations. Solve each of them as directed, in terms of the remaining letters:

20. $F = DgV$ for V; for D.

21. $E = \frac{1}{2}\,mv^2$ for m.

22. $r = \dfrac{RD}{d}$ for D; for R.

23. $E = iR + ir$ for R; for r.

24. $m = eit$ for e; for i; for t.

25. $W = \dfrac{Kbd^2}{L}$ for b.

24. Solving Verbal Problems by Equations. It has been pointed out that *the equation is the great tool for the solution of verbal problems.* The skilful use of this tool presupposes the ability to analyze the conditions of a verbal problem, and to state them in algebraic form. This work can be simplified to a great extent by following a definite plan. In particular, the following steps will be found helpful in the solution of verbal problems that involve the formation of simple equations:

1. Read the problem carefully. Try to *discover exactly what the problem calls for.* Represent the unknown quantity or missing number by a convenient letter. If the problem suggests the use of a definite *formula,* write that formula and identify each item of the formula in the problem.

2. When several unknown quantities are to be found, express all of them in terms of a *single* letter by means of the

relations stated in the problem. It is often advisable to make a *table* showing these relations.

3. Try to discover two expressions or quantities which are equal, and then write the resulting *equation* that represents the conditions of the problem.

4. *Solve this equation* and thus find the value of each missing number.

5. *Check* these results by means of the original statements of the problem.

NOTE. In checking the solution of a *problem*, it is necessary to use the original statements to see whether the solution "works." For the result obtained may correctly check the *equation*, but the equation may not correctly represent the conditions stated in the problem.

KEY WORDS: *missing numbers, relations, equation, solution, check.*

Illustrative Example. A carpenter and his helper built a garage in 12 da., at a total cost of \$560. The daily wage of the carpenter was three times that of his helper, and the cost of the building materials was \$320. How many dollars did each man earn per day?

Solution. 1. Let d = the number of dollars earned each day by the helper.

2. Then $3\,d$ = the number of dollars earned each day by the carpenter. Hence the total sum paid to both men in 12 da. was $12\,d + 36\,d$, or $48\,d$.

3. Hence $48\,d + 320 = 560$. (Explain.)

4. Solving this equation, we have

(1)	$48\,d + 320 = 560.$	Given.
(2)	$48\,d = 240.$	S_{320}.
(3)	$d = 5.$	D_{48}.

That is, the helper earned \$5 per day, and the carpenter earned \$15 per day.

5. Since $12 \times \$5 + 12 \times \$15 + \$320 = \$60 + \$180 + \$320 = \$560 =$ the total cost, the solution is correct.

Exercises. *Problems*

Number-Relation Problems

1. Four integers are represented by n, $n+5$, $2n-1$, $3n+2$. Their sum is 48. Find the numbers.

2. Divide 169 into two parts such that one part will be 25 less than the other part.

3. I am thinking of three integers whose sum is 84. The second is three times the first, and the third is four more than twice the second. Find the numbers.

4. The greater of two numbers is ten times the less, and their sum is 132. Find the numbers.

5. Can you find three consecutive integers whose sum is 108? Could the same sum have been obtained by adding three consecutive *even* integers? by adding three consecutive *odd* integers?

Suggestion. Consecutive integers may always be represented by n, $n+1$, $\cdots$, consecutive *even* integers by $2n$, $2n+2$, $\cdots$, and consecutive *odd* integers by $2n+1$, $2n+3$, $\cdots$.

Geometric Problems

6. A rectangular garden which is three times as long as it is wide requires 152 ft. of fencing. Find the length and the width of the garden.

7. A piece of lumber 16 ft. long is to be cut into four pieces such that each of three of them is 2 ft. longer than the preceding piece. Find the length of each piece.

8. The sum of the interior angles of a triangle equals 180°. Can a triangle be formed such that the numbers showing how many degrees each angle contains will be (1) consecutive integers? (2) consecutive even integers? (3) consecutive odd integers?

9. A wire model of a rectangular solid is to be made from a piece of wire 48 in. long. The height of this model is to be twice the width, and the length is to be 2 in. more than the height. Find the dimensions of the model.

10. The base of a triangle is 3 in. longer than twice its altitude. The area of the square constructed on the altitude is 12 sq. in. less than the area of the triangle. Find the base and altitude of the triangle.

Percentage Problems

11. The marked price of a chair is \$65. What was the cost of the chair if the marked price represents a profit of 25% on the cost?

12. In how many years will \$3500 yield an income of \$350, at 4% a year?

Suggestion. Use the formula $i = prt$.

13. What was the rate of interest on a note for \$450 if after $4\frac{1}{2}$ yr. at simple interest the total amount was \$551.25?

14. A 7% investment yielded annually \$16 more than a 6% investment. The sum of the two investments was \$4500. How much money was invested at each rate?

Solution. Let n = the number of dollars invested at 7%. Then $4500 - n$ = the number of dollars invested at 6%. The conditions of the problem may be tabulated as follows:

	Principal	Rate	Interest
First sum . . .	n	7%	$0.07\,n$
Second sum . .	$4500 - n$	6%	$0.06\,(4500 - n)$

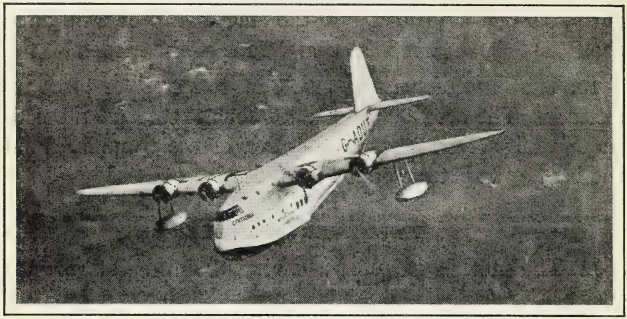
Hence the required equation is

$$0.07\,n = 0.06(4500 - n) + 16.$$

(Complete the solution.)

A

Motion Problems



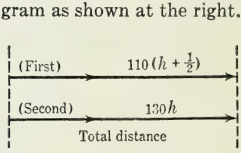
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15. Half an hour after an aviator, flying at the rate of 110 mi. an hour, started, another left from the same place to overtake him. How soon will the second aviator, at the rate of 130 mi. an hour, overtake the first?

Solution. Problems of this type are based upon the formula $d = rt$.

Let h = the number of hours the second aviator must fly to overtake the first. The conditions of the problem may be tabulated as shown at the left below, or pictured in a diagram as shown at the right.

Aviator	Rate (r)	Time (t)	Distance (d)
First . .	110	$h + \frac{1}{2}$	$110(h + \frac{1}{2})$
Second .	130	h	$130\,h$



Since the second aviator overtakes the first, the two distances are equal. Hence $130\,h = 110(h + \frac{1}{2})$.

Solving this equation, $h = 2\frac{3}{4}$.

That is, the second aviator must fly $2\frac{3}{4}$ hr. to overtake the first aviator.

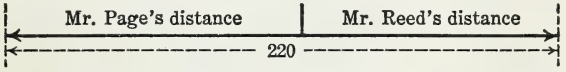
Check. The first aviator flies $3\frac{1}{4}$ hr. at the rate of 110 mi. an hour, and the second flies $2\frac{3}{4}$ hr. at 130 mi. an hour. Since $3\frac{1}{4} \times 110 = 357\frac{1}{2}$ and $2\frac{3}{4} \times 130 = 357\frac{1}{2}$, the solution is correct.

16. A train which averages 32 mi. an hour left Montreal for Toronto. It was followed 45 min. later on a parallel track by a fast passenger train, which overtook it in $1\frac{1}{2}$ hr. Find the rate at which the second train was travelling.

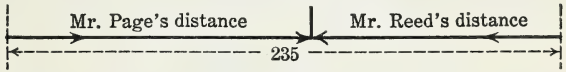
17. A tourist finds that he has to wait 4 hr. at a certain station before his next train leaves. He decides to take a walk into the suburbs and to return by trolley. If he walks at the rate of 3 mi. an hour, and if the trolley, which follows the same route, averages 12 mi. an hour, how much time may he spend walking?

18. Suppose that Mr. Page's average rate, when he travels by automobile, is 5 mi. per hour more than Mr. Reed's. Find the rate of each when:

(1) They start from the same place at the same time, but travel in opposite directions, and they are 220 mi. apart in 4 hr.



(2) They start toward each other at the same time from two places which are 235 mi. apart and meet in 5 hr.



(3) Mr. Page starts 1 hr. after Mr. Reed, from the same place and over the same road, and overtakes him in 6 hr.

(4) Mr. Page starts 2 hr. after Mr. Reed from a place 190 mi. distant. The two men travel toward each other and meet 4 hr. after Mr. Reed starts.

19. An airplane is travelling at the rate of r miles an hour. If a second airplane starts from the same place h hours later than the first, at what rate must it travel to overtake the first in n hours?

25. *Summary.* In this chapter you have learned

(1) How to classify the seven operations of algebra.

(2) The meaning and fundamental relations of positive and negative numbers.

(3) How to combine signed numbers by addition, by subtraction, by multiplication, and by division.

(4) The law of order and the law of grouping for addition.

(5) The laws of order, grouping, distribution, and exponents for multiplication.

(6) The law of distribution and the law of exponents for division.

(7) How to combine algebraic expressions by addition and subtraction.

(8) How to multiply an algebraic expression by a monomial or a polynomial.

(9) How to divide a monomial or a polynomial by a monomial, and a polynomial by a polynomial.

(10) The meaning and use of the rules of equality.

(11) How to transform an equation by addition or subtraction.

(12) How to solve an equation involving parentheses.

(13) How to solve an equation involving fractional coefficients.

(14) How to solve literal equations.

(15) How to solve verbal problems involving equations of the first degree in one unknown.

Test on Chapter I

1. Name the direct and inverse operations of algebra.

Perform the indicated operations in each case:

2. $32 - 4 \times 6 + 5 \times 3 - 2$.

3. $40 \times 10 \div 5 + 3$.

4. $5 \times 2^3 + 3 \times 4^3 - 5 \times 3$.

5. $60 \times 7 \div 10 - 8 \times 3^3 \div 9$.

6. How much larger is the sum of $2a + 3b - 5c$ and $3a - 5b + 2c$ than the difference found by subtracting the first polynomial from the second?

Find the product in each case:

7. $5 \times 2 \times (-3)^2$.
8. $(2\,xy)(-3\,x^2y^3)$.
9. $(4\,a^2b)(-3\,b)^2(-5\,a^2)$.
10. $x^{n+1} \cdot 2\,x^{n+2}$.

Remove parentheses and combine similar terms in each of the following expressions:

11. $a - a(1 - a) + a(a - 1) - (-1 - a)$.
12. $4\,x(-3 + 2\,x) - 7(6 - x^2) - [3 - 4(2 - x^2)]$.
13. Multiply $6 - 2\,a^2 + a^3 + 4\,a$ by $2\,a^2 - 3$.
14. Divide $8\,x + 5\,x^3 - 8\,x^2 - 3$ by $5\,x - 3$.

If x is a positive integer, determine whether y becomes larger or smaller in each of the following equalities when x is replaced by $3\,x$:

15. $y = 2\,x$.
16. $y = x^3$.
17. $y = \dfrac{1}{3\,x}$.
18. $y = \dfrac{1}{x^2}$.
19. $y = \dfrac{4\,x}{3}$.
20. $y = \dfrac{6}{x}$.

Solve each of the following equations for x:

21. $12\,x - 10 = 9\,x + 8 - 3\,x$.
22. $\dfrac{9\,x - 15\,c}{3} = \dfrac{7\,x + 5\,d}{4}$.
23. $5\,x - n = 2\,x + 9\,n$.
24. $\dfrac{x}{6} - 4 = \dfrac{5\,x}{18}$.

25. Determine by substitution whether or not $\frac{2}{3}$ is a root of the equation $6\,x + 5(3\,x - 1) = 15\,x - 1$.

26. In the equation $5\,x - 3\,n = nx + 7$, what is the value of n when $x = 4$?

27. The outside dimensions of a picture frame are w inches and $(3\,w - 7)$ inches. The frame is 2 in. wide. Write the expression which represents the area of the frame.

28. Solve for n the formula $l = a + (n - 1)d$.

First select the equation which correctly represents the conditions of the problem. Then solve the problem.

29. A man was hired by a farmer to work for 28 da. For each day he worked he was to receive \$4 and board. If he failed to work on any day, he was to be charged \$2 for his board. If he received \$94, how many days did he work?

(1) $2(28 - x) + 4x = 94$. (2) $4x - 2(28 - x) = 94$.
(3) $2(x - 28) - 4x = 94$.

30. A man sold his car for \$585, thus losing 35% of the original cost. How much did he pay for the car originally?

(1) $c - 35 = 585$. (2) $c + 0.35 = 585$. (3) $c - \frac{7c}{20} = 585$.

31. Write the equation which represents the conditions of the following problem: A man invests \$1500, a part at 5% and the remainder at 4%. His annual income from these investments is \$68.70. How many dollars are invested at each rate?

32. On a certain building lot, the house, the garage, the drive, and the walk together occupy 1600 sq. ft. of space. Three fourths of the remaining space is used for a lawn, and the rest for a garden. If the lawn covers 3600 sq. ft., what is the area of the entire lot?

Why are *wise* few, *fools* numerous in the excesse?
'Cause, wanting *number*, they are *numberlesse*.—LOVELACE

Anyone who understands algebraic notation reads at a glance in an equation results reached arithmetically only with great labor and pains.—A. COURNOT

CHAPTER II

FUNCTIONS OF THE FIRST DEGREE IN TWO OR THREE VARIABLES

1. The Graph of a Function

26. Constants and Variables. In mathematics, numbers are represented by means of symbols. Any mark that is used to represent a number is called a numerical symbol, as, for example, $1, -7, 0, +8, a, b, c, x, y, z, \cdots$.

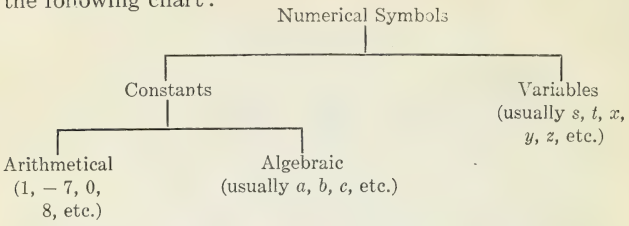
1. *Numerical symbols are either arithmetical or algebraic.* If the symbol represents a known number, it is an *arithmetical symbol,* but if it represents an unknown number, it is an *algebraic symbol.* Thus, $1, -7, 0, +8$ are arithmetical symbols, but a, b, c, x, y, z are algebraic symbols. The number which a numerical symbol represents is called the *value* of the symbol.

2. *Numerical symbols are either variables or constants.* If the value of a numerical symbol is *fixed,* the symbol is called a *constant.* If the numerical symbol has a *changing* value, the symbol is called a *variable.*

You are familiar with the formula for the area (A) of a circle, $A = \pi r^2$. Here the value of π is fixed, but r, the radius of the circle, is different for different circles. π is therefore called a constant, and r is a variable.

You will readily see that all arithmetical symbols are constants, while an algebraic symbol may be either a constant or a variable. It has become more or less customary to use the first letters of the alphabet, $a, b, c, \cdots$, to represent constants, and the last letters, $\cdots, w, x, y, z$ for variables.

There is, however, no hard-and-fast rule about this. Study the following chart:



27. Variables depending upon One Another: Functions. If x represents the number of inches in the side of a square, then x^2 represents the number of square inches in the area. If you imagine the square to be changing in size, you will see that both x and x^2 are variables. It is apparent that the value of x^2 *depends* upon the value of x; that is, if we assume a value for x, then x^2 has a corresponding value which can be computed. If we picture the value of x^2 as depending upon the value of x, we call x the *independent variable*, and x^2 the *function* of x. You must note that the function x^2 is merely a variable whose value depends upon the value of the first variable x. The function x^2 is also called the *dependent variable*.

It is clear that we might assume a value for x^2 and then compute the value of x. In this case x^2 would be the independent variable while x would be the function or dependent variable.

In the formula $A = \pi r^2$, r is the independent variable and A is the dependent variable. We could also say that A or πr^2 is a function of r.

We are now ready for this definition:

If two variables, such as x and y, are so related that to each value of x (the independent variable) there corresponds a definite value or set of values of y (the dependent variable), y is called a function of x.

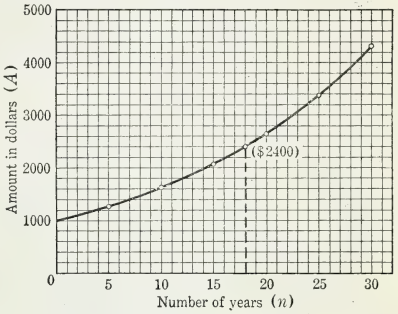
Thus, if $y = 3x + 6$, we may assign different values to x and compute the corresponding values of y. Then y is a function of x. Also, if $v = 6s$, p is a function of s.

In picturing a formula, or a function, it is customary to plot values of the independent variable on the *horizontal* axis, while values of the dependent variable are shown on the *vertical* axis.

For example, the diagram at the right gives us a picture of the variation of the amount (A) in dollars of a sum of \$1000 at the end of a certain number (n) of years when interest is compounded at 5%.

The independent variable (n) is represented on the horizontal axis, and the dependent variable (A) on the vertical axis. The values of the function A for various values of n are shown by points (indicated by small circles).

The simplest curve which passes through these points is called a *smooth curve*. By means of this curve it is possible to estimate other values of A for intermediate values of n. Thus, when $n = 18$, $A = 2400$, approximately. Hence the amount of \$1000 at the end of 18 yr., compounded at 5%, is approximately \$2400.

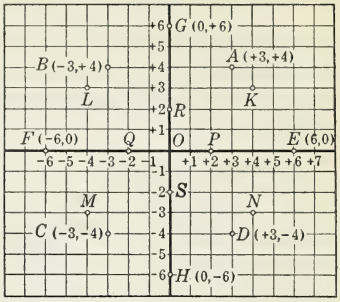


Growth of \$1000 at 5% compound interest

In this chapter you will learn how to draw and interpret the graphs of simple functions. This knowledge is of great importance in mathematics and science.

28. Locating Points. In the figure on page 58 we have two number scales, or axes, one horizontal and one vertical, with their zero points coinciding. This point of coincidence is called the *origin*.

It is easy to see that we can find a point if we know how it is situated with respect to the axes. Thus, *counting from the origin*, the point A is 3 units to the right and 4 units up. The two numbers $+3$ and $+4$, which show the position of the point A, are called the *coördinates* of the point. The horizontal coördinate, $+3$, is called the *abscissa* of the point, and the vertical coördinate, $+4$, is called the *ordinate* of the point. Explain the meaning of "horizontal" and "vertical."



It is customary to write the coördinates as shown beside the letter A in the figure. Note that we write the abscissa first. We read the coördinates "three, four," and we call $A(+3, +4)$ "the point three, four."

From the arithmetical point of view the coördinates are *numbers*, but from the standpoint of geometry they are directions and distances. Distances to the *right* of the vertical axis and those *above* the horizontal axis are regarded as *positive*, while distances to the *left* of the vertical axis and those *below* the horizontal axis are regarded as *negative*.

Now examine the point B. We may locate this point by counting (from the origin) 3 units to the left and 4 up. Here the abscissa is -3 and the ordinate is $+4$. We read the coördinates "minus three, plus four" and the point B is named that way. Similarly, we can read and interpret the other points in the diagram.

When we locate a point by the use of its coördinates, as shown above, we are said to *plot* the point.

Exercises. *Locating Points*

1. Read the coördinates of the points K, L, M, N, P, Q, R, S, O in the figure on page 58.

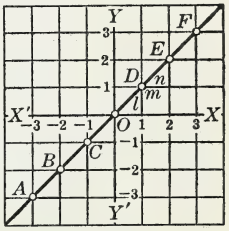
2. Plot the points (2, 6), (− 4, 3), (− 1, − 4), (5, − 2), (5, 0), (− 5, 0), (0, 5), (0, − 5).

3. Locate the points $A(4, 2)$; $B(-4, 2)$; $C(-4, -6)$; $D(4, -6)$. Connect the four points in succession.

(1) What kind of figure is $ABCD$?

(2) Draw the diagonals AC and BD, and call their point of intersection P. What are the coördinates of P?

4. Plot the points: (− 3, − 3), (− 2, − 2), (− 1, − 1), (0, 0), (1, 1), (2, 2), (3, 3). With the aid of geometric facts which you have learned, try to *prove* that they lie on the same straight line.



SUGGESTION. The triangles shown in the figure are evidently congruent. (Why?) Prove that the sum of the three angles l, m, and n at D is 180°; do the same for the three angles at A, B, C, O, E, and F.

5. Plot the following points: (− 6, − 3), (− 4, − 2), (− 2, − 1), (0, 0), (2, 1), (4, 2), (6, 3). Connect these points in succession. Does the resulting graph appear to be a broken or straight line? Can you give a geometric reason?

6. (1) Locate these points: $A(3, 8)$, $B(1, 6)$, $C(-1, 4)$. Connect them in succession. What do you observe?

(2) Using the same axes, locate the points $D(6, -1)$, $E(2, 1)$, and $F(0, 2)$. Connect them in succession.

(3) Extend the lines AC and DF until they intersect, and give the coördinates of their point of intersection.

29. Plotting the Graph of a Function. We have seen that if x represents the number of inches in the side of a square, x^2 represents the number of square inches in the area. We have also seen that if the square is changing in size, x and x^2 are variables depending on one another, so that x^2 is a function of x. The letter y is sometimes used to represent the function, in which case we may write the *equation* $y = x^2$.* In such an instance, then, x is the variable and y, or x^2, the function.

Let us see if we can draw a picture to show how the function y depends on the variable x. We shall find that we can do this by using the method of locating points. Let us agree to measure values of the variable x horizontally and the values of the function y, or x^2, vertically.

If we give x a value, say $+2$, the corresponding value of y is $+4$. Then the point $(+2, +4)$ represents this pair of values, the abscissa $+2$ representing a value of the variable x, and the ordinate $+4$ the corresponding value of the function y, or x^2. It is clear that in a similar manner every pair of corresponding values of the variable and function can be represented by a point.

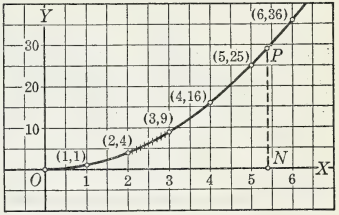
When we use the axes for this purpose the horizontal axis is known as the x-axis, or axis of the variable, and the vertical axis is known as the y-axis, or axis of the function. Distances measured horizontally from the y-axis measure values of x and distances measured vertically from the x-axis measure values of y. The point $(-2, 4)$ may be located by counting from the origin 2 units to the left and 4 units up.

* Equations like this, although not of the first degree, are introduced here because they are excellent illustrations of the function idea. They will be discussed in more detail at a later point.

Note the following tabulation of values of the variable x and corresponding values of the function y, or x^2, and the points corresponding to these pairs of values:

Variable	x	0	1	2	3	4	5	6	...
Function	x^2, or y	0	1	4	9	16	25	36	...

In drawing the graph at the right we have used different units of measurement for the variable and function merely for convenience in placing the points in the space at our disposal.



We may, of course, use fractional values, as shown in the following table:

Variable	x	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3
Function	x^2, or y	4	4.41	4.84	5.29	5.76	6.25	6.76	7.29	7.84	8.41	9

These pairs of values give nine new points between (2, 4) and (3, 9). How many more such points can we find between (2, 4) and (3, 9)?

NOTE. In computations involving decimals the following statements will be found helpful:

1. Most measurements are only approximately correct. Moreover, decimal results based on approximate values cannot be made more accurate than the least accurate value used.

2. In "rounding off" decimals to any desired degree of accuracy, (1) if the figure which is dropped is *greater than* 5, the preceding figure should be *increased* by 1; (2) if the figure which is dropped is *less than* 5, the preceding figure remains *unchanged*; (3) if the figure which is dropped is 5, the preceding figure is *increased* by 1 when it is *odd*, but remains *unchanged* when it is *even* or *zero*.

If we draw through the points as smooth a line as we can, we obtain a *curve* which pictures the way the value of the function depends on the value of the variable. If we take any point P *on the curve*, the abscissa ON represents a value of the variable, and the ordinate NP represents the corresponding value of the function. This curve is called the *graph of the function* y, or x^2. When we sketch this curve we plot the function x^2, or y, against the variable x, or the dependent variable x^2, or y, against the independent variable x. For some purposes we say the curve is the graph of the equation $y = x^2$ in the two variables x and y.

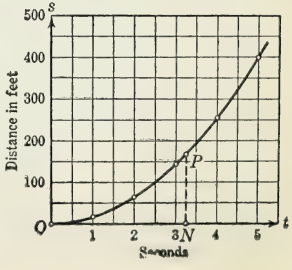
Here is an example of another graph of this type:

If a body be allowed to fall toward the earth from a state of rest, it is found that it falls approximately $16\,t^2$ feet in t seconds. If we represent $16\,t^2$ by s, we have $s = 16\,t^2$.

Note the following tabulation of pairs of corresponding values of variable and function:

Variable	Independent Variable	Time	t	0	1	2	3	4	5	···
Function	Dependent Variable	Distance	s, or $16\,t^2$	0	16	64	144	256	400	···

Plotting points and sketching a curve as before, we get the graph at the right:



Note that again we choose units of measurement to enable us conveniently to place the curve in the space at our disposal. In cases where it is convenient to use equal units on both axes it is advisable to do so.

If on the graph we take any point P, the abscissa ON represents a number of seconds, and the ordinate NP the distance the body falls in that number of seconds. For example, if $ON = 3.25$, $NP = 169$; that is, the body falls 169 feet in 3.25 seconds.

You should now practise making similar readings from the graph and check by substituting in the function.

Exercises. *Graphs of Functions*

1. From the graph on page 61 estimate the values of the function corresponding to the following values of the independent variable: 1.5, 1.75, 3.5, 4.25, $5\frac{1}{2}$, 6.75.

2. In Exercise 1, check your estimates by squaring the variable.

3. Why do you think we considered only positive values in Exercise 1?

NOTE. From the algebraic point of view we could, of course, use negative values of the variable. Thus, if $x = -3$, $x^2 = +9$. This pair of values would be represented by the point $(-3, +9)$.

4. (1) Construct the graph shown on page 57 on a larger scale; for example, let 1 in. represent \$1000 and also 5 yr. Use the following approximate data:

n	0	5	10	15	20	25	30
A	1000	1275	1630	2080	2655	3385	4320

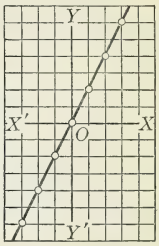
(2) How does the graph show that A is doubled approximately every 14 yr.?

(3) From the graph find what \$1000 will amount to at the end of 23 yr. when compounded at 5%.

2. Graphs of Linear Functions

30. Picturing Linear Functions. Suppose that we want to draw the graph of the function $2x$ for values of x between -3 and $+3$. We select convenient values of x and compute the corresponding values of $2x$, which we denote by y. Thus we obtain a table of values such as the following:

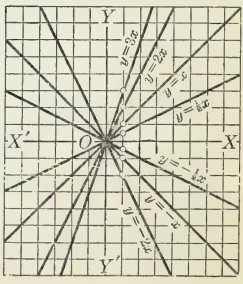
$x =$	0	1	2	3	-1	-2	-3
$y =$	0	2	4	6	-2	-4	-6



The diagram at the right shows that the points plotted from the pairs of values given in the table appear to lie on a straight line, which is the picture of the function $2x$, or of the equation $y = 2x$. Hence $2x$ is said to be a *linear* function of x, and the equation $y = 2x$ is called a *linear equation*. Observe that the graph passes through the origin (0, 0) and the point (1, 2).

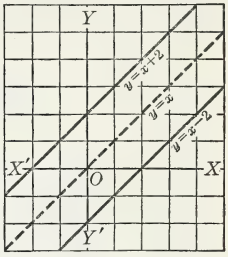
NOTE. The symbol y is frequently used to denote the function of x.

1. *Functions of the Type mx.* Using the method described above, we may readily plot the graph of *any* equation of the type $y = mx$, where m may be either a positive number or a negative number. This diagram shows the "family," or "system," of graphs corresponding to definite values assigned to m in the equation $y = mx$. Observe particularly the graph of $y = x$ and of $y = -x$.

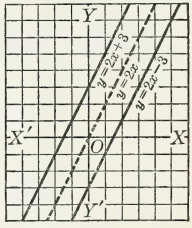


It can be shown that *the graph of any equation of the type* $y = mx$ is

a straight line. Each of these graphs evidently passes through the origin and also through the point $(1, m)$. Moreover, it is readily seen that as the absolute value of m increases, the graph of $y = mx$ is "*rotated*" toward the y-axis, and as the absolute value of m decreases, the graph is rotated toward the x-axis.



2. *Functions of the Type* $x + b$. If we picture an equation such as $y = x + 2$, using the method explained above, we find that the resulting graph is also a straight line. In like manner, the graph of any equation of the type of $y = x + b$ is a straight line. By assigning definite numerical values to b, we obtain the "family," or "system," of graphs shown in the diagram above. Observe that these graphs are *parallel* straight lines. It is exactly as though the graph of $y = x$ had been shifted or "translated" to a new position, the value of the constant number b indicating the point at which the graph crosses the y-axis.



3. *Functions of the Type* $mx + b$. By combining the methods suggested above, it is evidently possible to sketch at once the graph of an equation such as $y = 2x + 3$. It is merely necessary (1) to "rotate" the graph of $y = x$ until it corresponds to the graph of $y = 2x$, and (2) to "translate" the resulting graph until it cuts the y-axis at the point $y = 3$. A similar plan may be followed in the case of any other equation of the type $y = mx + b$. *The required graph is always found to be a straight line.*

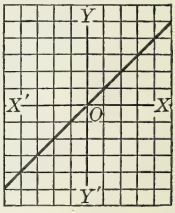
Hence any expression of the type $mx + b$ is called a *linear function* of x, and any equation of the type $y = mx + b$ is called a *linear equation* in two variables, x and y.

To draw the graph of a linear equation *only two of its points are necessary.* It is generally advisable, however, to plot *three* points whose coördinates can readily be located.

Exercises. *Plotting Graphs of Linear Equations*

1. Draw the graph of the equation $y = x$.

Solution. In this case, to every value of x there corresponds an *equal* value of y. Thus, when $x = 1$, $y = 1$; when $x = -2$, $y = -2$, and so on. Hence the graph forms an angle of 45° with either axis. (Why?)



2. Using the same axes, draw the graphs of the following equations: (1) $y = 3x$; (2) $y = 4x$; (3) $y = 5x$.

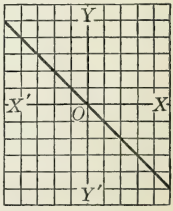
3. Using the same axes, draw the graphs of the following equations:

(1) $y = \frac{3}{4}x$; (2) $y = \frac{1}{2}x$; (3) $y = \frac{1}{4}x$; (4) $y = \frac{2}{3}x$.

SUGGESTION. In the case of $y = \frac{3}{4}x$, observe that when $x = 4$, $y = 3$.

4. Draw the graph of the equation $y = -x$.

SUGGESTION. When $x = 0$, $y = 0$; when $x = 1$, $y = -1$; when $x = -1$, $y = 1$, and so on. The resulting graph is shown in this figure.



5. Using the same axes, draw the graphs of the following equations: (1) $y = -2x$; (2) $y = -3x$; (3) $y = -4x$; (4) $y = -5x$.

6. Using the same axes, draw the graphs of the following equations: (1) $y = -\frac{2}{3}x$; (2) $y = -\frac{1}{3}x$; (3) $y = -\frac{2}{5}x$; (4) $y = -\frac{1}{5}x$.

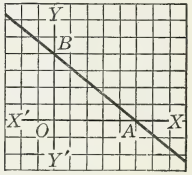
7. Draw the graph of each of the following equations: (1) $y = x + 3$; (2) $y = x + 4$; (3) $y = x - 3$; (4) $y = x - 4$.

Draw the graph of each of the following equations:

8. $y = 2x + 1$. **10.** $y = 2x - 5$. **12.** $y = \frac{1}{2}x + 4$.

9. $y = 3x + 2$. **11.** $y = 3x - 2$. **13.** $y = \frac{1}{2}x - 4$.

31. Intercepts. This diagram shows the graph of the equation $y = -\frac{4}{5}x + 4$, or $4x + 5y = 20$. The distances OA and OB which are cut off on the axes by the graph are called *intercepts*. OA is the *x-intercept* and OB is the *y-intercept*.



Since $x = 0$ for every point on the y-axis, we can find the y-intercept by putting $x = 0$ and solving for y. Thus in the equation above, when $x = 0$, $y = 4$. That is, $OB = 4$. Similarly, to find the x-intercept put $y = 0$ and solve for x. Thus, when $y = 0$, we have $0 = -\frac{4}{5}x + 4$, or $x = 5$. Hence $OA = 5$.

In like manner the y-intercept of the equation $y = mx + b$ is b, and the x-intercept is $-\frac{b}{m}$.

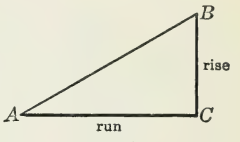
Since two points determine a straight line, we can draw the graph of a linear equation of the type $y = mx + b$ after we have found its intercepts. This method is obviously impossible when the equation is of the type $y = mx$. (Why?)

Exercises. *Using Intercepts in Plotting Graphs*

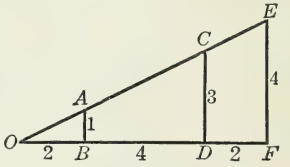
Find the intercepts for each of the following equations, and plot the graph:

1. $y = 2x + 6$.
2. $y = 3x - 6$.
3. $y = -2x + 10$.
4. $y = \frac{1}{2}x + 4$.
5. $y = -3x - 9$.
6. $y = 4x - 5$.
7. $2y = 6 - 3x$.
8. $4x = 3y + 12$.
9. $5x - 2y = 10$.

32. Slope of a Line.* If a straight line rises 6 feet in a horizontal distance of 100 feet, its *grade*, or *slope*, is said to be 6 in 100, or $\frac{6}{100}$, or 6%. That is, the slope of a straight line AB is the ratio of its vertical *rise* to the corresponding horizontal distance, or *run*. It is thus apparent that the slope of a line is a measure of its "steepness."



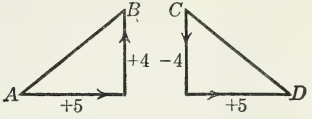
In the diagram below, the slope of OA is $\frac{1}{2}$, or 0.5. Now the triangles OAB, OCD, and OEF are similar. (Why?) Hence it is evident that as OA is extended, the slope continues to be the same. Thus, the slope of OC is $\frac{3}{6}$ or $\frac{1}{2}$, and the slope of OE is $\frac{4}{8}$ or $\frac{1}{2}$. That is, *the slope of a straight line is the same throughout its entire extent.*



33. Positive and Negative Slope. The figures on the next page will serve to show that a slope may be positive or negative. Thus, the slope of AB is $\frac{+4}{+5}$, or $\frac{4}{5}$, while the slope of CD is $\frac{-4}{+5}$, or $-\frac{4}{5}$.

* Sections 32–34, though very desirable in a Second Course, may be regarded as optional.

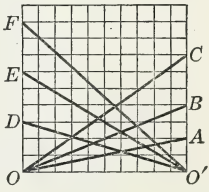
It will be seen at once that the slope of AB is *positive* whether we go from A to B or from B to A. Likewise, the slope of CD is *negative* whether we go from C to D or from D to C. That is, the slope of a line which *rises* to the right is *positive*, while the slope of a line which *falls* to the right is *negative*.



Exercises. *Slopes of Lines*

1. In this figure, what is the slope of OA? of OB? of OC?

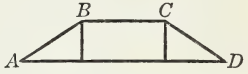
2. In the figure of Exercise 1, what is the slope of $O'D$? of $O'E$? of $O'F$?



3. On squared paper draw the line AB determined by the points A (2, 1) and B (6, 5). What is the slope of the line AB?

4. A line passes through the points C (1, 6) and D (11, 1). What is the slope of the line CD?

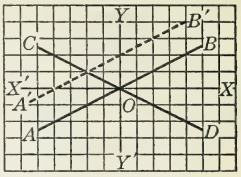
5. This diagram represents a bridge and its approaches. If an automobile crosses the bridge, either from A to D or from D to A, show that the slope of AB is positive, and the slope of CD is negative. Show that the slope of BC is zero.



6. Draw a line AB whose slope is (1) $\frac{1}{3}$; (2) $\frac{3}{4}$; (3) $\frac{5}{6}$; (4) 2 : 1.

7. Draw a line CD whose slope is (1) $-\frac{1}{2}$; (2) $-\frac{2}{3}$; (3) $-\frac{9}{10}$; (4) $-3 : 1$

34. Linear Equations and the Slopes of their Graphs. It follows, from the explanations given above, that the slope of the line AB in this diagram is $\frac{1}{2}$.



Likewise, the slope of the line $A'B'$, which is parallel to AB, is $\frac{1}{2}$.

That is, every point on AB is such that its ordinate, or y value, is $\frac{1}{2}$ of the corresponding abscissa, or x value. This fact may be indicated by writing $\frac{y}{x} = \frac{1}{2}$, or $y = \frac{1}{2}x$. Hence AB is the graph of the equation $y = \frac{1}{2}x$. It appears, therefore, that *the slope of the graph is the coefficient of* x, namely, $\frac{1}{2}$.

Similarly, it is evident that the slope of CD is $-\frac{1}{2}$, and that CD is the graph of the equation $y = -\frac{1}{2}x$. The slope in this case is also the coefficient of x, namely, $-\frac{1}{2}$.

In general, the graph of any linear equation of the type $y = mx$ has for its slope m, the coefficient of x. This fact makes it possible to construct the graph of such a linear equation at once, without a table of values. (See Fig. 1.)

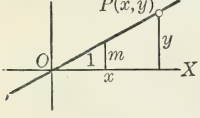


Fig. 1

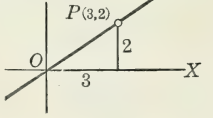


Fig. 2

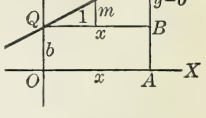


Fig. 3

Thus, the graph of $y = \frac{2}{3}x$ (see Fig. 2) is determined by the origin O and a point P whose coördinates are (3, 2).

Now consider an equation of the type $y = mx + b$. Such an equation may be written in the form $y - b = mx$, or

$\frac{y-b}{x}=\frac{m}{1}$. Fig. 3 pictures this relation. If the coördinates of any point P on the graph are (x, y), then $PA = y$ and $PB = y - b$. Hence in this case also the slope of the graph is indicated by m, the coefficient of x.

Exercises. *Using the Slope in Plotting Graphs*

Draw the graph of each equation, using the slope method:

1. $y = 3x$.	**5.** $y = 4x + 1$.	**9.** $y = -3x + 4$.
2. $y = 10x$.	**6.** $y = 4x - 1$.	**10.** $y = -3x - 4$.
3. $y = \frac{1}{4}x$.	**7.** $y = \frac{1}{2}x + 2$.	**11.** $y = -\frac{1}{5}x + 3$.
4. $y = \frac{1}{10}x$.	**8.** $y = \frac{1}{2}x - \frac{1}{2}$.	**12.** $y = -\frac{1}{5}x - 2\frac{1}{2}$.

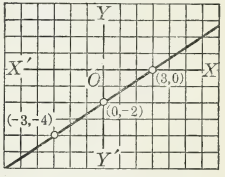
13. Draw the graph of the equation $x - y = 5$.

SUGGESTION. The equation may be written in the form $y = x - 5$.

14. Draw the graph of the equation $2x - 3y = 6$.

Solution 1. *Using a Table.* By assigning convenient values to x, we may obtain a *table* of corresponding values such as the following:

If $x =$	-3	0	3
then $y =$	-4	-2	0



These pairs of values enable us to draw the required graph, as shown in the figure.

Solution 2. *Using Intercepts.* When $x = 0$, $y = -2$. Hence the y-intercept is -2. When $y = 0$, $x = 3$. Hence the x-intercept is 3. We may therefore draw the line determined by these intercepts.

Solution 3. *Using the Slope.* The given equation may be transformed so that it corresponds to the form $y = mx + b$. Hence we find that $y = \frac{2x-6}{3}$, or $y = \frac{2}{3}x - 2$. That is, the slope is $\frac{2}{3}$ and the y-intercept is -2.

Draw the graph of each of the following equations, using any convenient method:

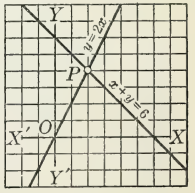
15. $x + y = 3$.
16. $x - y = 4$.
17. $x - 2y = 6$.
18. $y - x = 6$.
19. $2x + 3y = 6$.
20. $3x - 6y = 12$.
21. $3x + y = 0$.
22. $5x - y = 0$.
23. $x + 3y = 6$.
24. $4x - 5y = 15$.
25. $7x + 2y = 10$.
26. $0.1x + 0.2y = 1$.

27. Each of the following pairs of coördinates represents a point on the graph of one of the equations in Exercises 19–26 above. Select the correct equation in each case:

a. (3, 0). *b.* (1, −3). *c.* (5, 1). *d.* (8, 2). *e.* (2, 10). *f.* (3, 1). *g.* (2, −2). *h.* (4, 3).

35. Simultaneous Linear Equations. The graph of a linear equation evidently pictures a definite relationship between two variables. This fact makes it possible to use the graphic method of representing linear equations as an important tool in the solution of mathematical problems involving *two unknowns*. A simple verbal problem will furnish a convenient illustration.

Illustrative Example. Find two numbers such that one is twice as large as the other and their sum is 6.



Solution. If the smaller number is represented by x, while y represents the larger number, the conditions of the problem are stated by the two equations

(1) $x + y = 6$,

(2) $y = 2x$.

The graphs of these linear equations are shown in the diagram. Obviously every point on the graph of $x + y = 6$ is such that the sum of its abscissa and its ordinate equals 6. Again, every point on the graph of $y = 2x$ is such that its ordinate is twice the corresponding abscissa.

Now observe that the two graphs *intersect* at the point P, whose coördinates are $x = 2$ and $y = 4$. Hence it is clear that these values of x and y satisfy *both* equations. Then, since the two straight lines have no other point in common, we infer that (2, 4) is the only pair of numbers which will satisfy both equations. Moreover, the numbers 2 and 4 meet the original conditions of the given verbal problem. We have therefore solved the given problem *graphically*.

When we picture a linear equation such as $x + y = 6$, we find that the coördinates of *any* point on its graph satisfy the equation. Every such pair of values of x and y is called *a solution of the equation*. That is, *a single linear equation containing two unknowns has an indefinite number of solutions.* Hence such an equation is called an *indeterminate equation.*

If, however, the graphs of two linear equations in x and y intersect, as in the preceding example, it is evident that they are satisfied by the same pair of values of x and y. Such equations are therefore said to be *simultaneous.*

To solve a pair of simultaneous linear equations is to find their common solution.

NOTE. The explanations which are given above can be extended to include the solution of higher types of simultaneous equations. There are indeterminate equations, and also simultaneous equations, which are not linear. For example, we may be concerned with higher powers of the unknowns.

Exercises. *Solving Pairs of Linear Equations by Graphs*

Solve graphically each of the following pairs of equations:

1. $x + y = 5$, $x - y = 1$.

2. $x + y = 7$, $x - y = 3$.

3. $3x + y = 6$, $y = 3x$.

4. $x + 2y = 8$, $y = \frac{1}{2}x$.

5. $x + 4y = 2$, $x - y = -3$.

6. $y - x = 1$, $3y = 2x$.

7. $3x + 4y = 2$, $x - 2y = 9$.

8. $4x - 3y = -1$, $y + 2 = 2x$.

9. $4x + 3y = 9$, $3y = 8x$.

NOTE. When the graphs of two linear equations are *parallel*, the equations do not have a common solution. In such cases both equations, when written in the form $y = mx + b$, have the same value of m. Thus, the equations $3x - 2y = 6$ and $6x - 4y = 8$ may be written in the form $y = \frac{3}{2}x - 3$ and $y = \frac{3}{2}x - 2$, respectively. Such equations are called *inconsistent equations*.

Again, one equation may be merely a different way of writing another equation. Thus, $2x + 4y = 2$ and $4x + 8y = 4$ can be reduced to the same equation, namely, $x + 2y = 1$. Such equations are therefore represented graphically by *the same straight line*. They are called *dependent equations*, because each can be derived from the other. Equations which have distinct graphs, thus showing that they are not dependent on each other, are called *independent equations*.

In each of the following cases, state whether or not the given equations have a common solution, and give your reason:

10. $x + y = 4$, $x + y = 3$.

12. $5x - 3y = 2$, $10x - 6y = 4$.

14. $x + 2y = 8$, $2x - 3y = 4$.

11. $3x + 4y = 9$, $3x + 2y = 9$.

13. $9x - 6y = 15$, $3x - 2y = 5$.

15. $0.4x + 1.8y = 6$, $0.2x + 0.9y = 5$.

Solve each of the following problems graphically:

16. Find two numbers whose sum is 8, such that one is three times as large as the other.

17. The greater of two numbers is five times the less, and their difference is 4. Find the numbers.

18. Find two consecutive integers whose sum is 11.

19. The length of a rectangular flower bed is to be four times the width. If 20 ft. of wire fencing are to be used for inclosing it, how long can the bed be made?

HISTORICAL NOTE. René Descartes (1596–1650), a French mathematician, philosopher, and physicist, was chiefly responsible for the development of the method of plotting points by means of coördinates and of investigating the properties of curves. According to one anecdote, this idea first came to him as he watched a fly crawling along the ceiling

of his room, near a corner, and he tried to develop a means of expressing the fly's motion in terms of its distance from each of the walls. Regardless of how the original inspiration came, however, Descartes's idea of finding an algebraic equation representing a given curve (graph) is one of the greatest contributions ever made to mathematics.

3. Algebraic Solution of Sets of Equations of the First Degree

36. Solving Sets of Equations containing Two Unknowns. The graphic method of solving sets of linear equations is not always convenient or accurate. The solution may involve large numbers or fractions, and hence the results in such a case can be obtained only approximately on squared paper. Some other method is therefore desirable.

We are able to solve algebraically an equation of the first degree in one unknown. Accordingly, if we can combine a set of equations containing two unknowns in such a way that one of the unknowns will disappear, the required solution becomes possible. This process is called *elimination*, and may be accomplished by either of the two methods described below.

1. *Elimination by Addition or Subtraction.* In a pair of linear equations, when the coefficients of one of the unknowns are the *same* in absolute value, we may eliminate that unknown either by adding the equations or by subtracting one from the other.

When the coefficients of the unknown which is to be eliminated are *not* the same in absolute value, it becomes necessary to make these coefficients alike. This may be done by a single or a repeated application of the multiplication axiom. It is then possible to eliminate this particular unknown either by addition or by subtraction.

Illustrative Example. Solve the pair of linear equations

$$(1) \quad x - 2y = 8,$$
$$(2) \quad 7x + 3y = 5.$$

Solution. To eliminate y, we multiply both members of the first equation by 3, and both members of the second equation by 2. The solution may be arranged in brief form as follows:

(1) $x - 2y = 8.$	Given	Substituting 2 for x in (1),		
(2) $7x + 3y = 5.$	Given			
(3) $3x - 6y = 24$	$M_{(1) \text{ by } 3}$.	(7) $2 - 2y = 8.$		
(4) $14x + 6y = 10$	$M_{(2) \text{ by } 2}$.	(8) $2y = -6.$		Tr.
(5) $17x = 34$	Adding	(9) $y = -3.$		D_2.
(6) $x = 2.$	D_{17}.			

Hence $x = 2$ and $y = -3$.

Check. Substituting 2 for x and -3 for y in each equation,

(1) L.S. $= 2 + 6 = 8$; R.S. $= 8$. | (2) L.S. $= 14 - 9 = 5$; R.S. $= 5$.

$\therefore$ The solution $x = 2$, $y = -3$ satisfies both of the given equations.

2. *Elimination by Substitution.* In this case we express one unknown *in terms of the other*, from one of the given equations, and substitute for that unknown in the other equation. The solution can then be completed in the usual way

Illustrative Example. Solve the pair of equations

$$(1) \quad 3x + 5y = 25,$$
$$(2) \quad x - 4y = -3.$$

Solution. Solving equation (2) for x in terms of y, we obtain

$$x = 4y - 3.$$

Substituting this value of x in equation (1), we have

$$3(4y - 3) + 5y = 25.$$

Then $$12y - 9 + 5y = 25. \quad \text{(Why?)}$$

That is, $$17y = 34;$$

whence $$y = 2. \quad \text{(Why?)}$$

Substituting 2 for y in equation (2), we have

$$x - 8 = -3,$$

or

$$x = 5.$$

Hence the required solution is $x = 5$ and $y = 2$.

Check. Use the method shown on page 76.

Try to summarize in your own words the two methods described above. The exercises which follow will suggest that in certain cases the substitution method is preferable.

Exercises. *Solving Pairs of Equations Algebraically*

Solve the following pairs of equations by the method of addition or subtraction. Check each set of results.

1. $2x + y = 11,$
   $3x - y = 9.$

2. $m + 5n = 7,$
   $m - 3n = -1.$

3. $r - 3s = 4,$
   $5r + 3s = -1.$

4. $4a + 3b = -4,$
   $6a + 5b = -7.$

5. $0.3x + 0.7y = 5,$
   $0.4x + 0.3y = 3.5.$

6. $\frac{1}{4}c + 3d = 2\frac{1}{2},$
   $3d - \frac{3}{4}c = \frac{1}{2}.$

Solve the following pairs of equations by the method of substitution. Check each set of results.

7. $x = 2y - 1,$
   $4x - y = 10.$

8. $s = 2r - 2,$
   $r + 2s = 11.$

9. $a - 2b = 1,$
   $2a + b = 12.$

10. $4m + n = 9,$
    $3m - 5n = 1.$

11. $2p - 8q = 4,$
    $5p - 12q = 14.$

12. $3x = 3y + 6,$
    $4x - 5y = 3.$

13. Using the formulas $i = prt$ and $A = p + i$, obtain a formula for A which does not contain i.

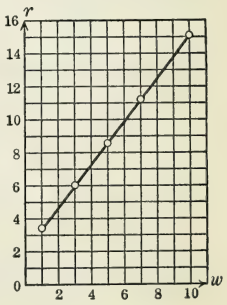
14. In the formulas $l = a + (n - 1)d$ and $s = \frac{1}{2}n(a + l)$, eliminate l by substitution and thus obtain a formula for s which does not contain l.

37. Making Formulas based on Tables. The experiments on which most technical formulas are based are usually recorded in the form of tables. The numbers in such a table tell the story of the experiment. Formulas may often be obtained from the underlying tables by inspection. When the given data are more complicated, however, it is necessary to use more accurate methods. In particular, we shall be able to use the principles developed in the preceding pages in the case of any tables that can be represented graphically by a straight line.

Illustrative Example. An experiment involving two related variables, w and r, yielded the following table of corresponding values. What is the formula connecting w and r?

$w =$	1	3	5	7	10
$r =$	3.4	6	8.6	11.2	15.1

Solution. 1. We draw a graph picturing the table. This graph appears to be a straight line.



2. Consequently the required formula must be of the type $y = mx + b$. That is, $r = mw + b$. In this formula, m and b are unknown.

3. Since the given table contains related values of w and r, we can substitute two sets of values from this table in the formula $r = mw + b$. Thus, using the values $w = 1$ and $r = 3.4$, we have $3.4 = m + b$. Again, from the values $w = 3$ and $r = 6$ we have $6 = 3\,m + b$.

4. Solving this pair of equations for m and b, we find that $m = 1.3$ and $b = 2.1$. Hence the required formula is $r = 1.3\,w + 2.1$.

5. We must test the correctness of this formula by substituting other corresponding sets of values for w and r. It will be found that the other given sets of values also satisfy this formula.

Exercises. *Obtaining Formulas from Tables*

Following the method suggested on the preceding page, try to discover the formula suggested in each case:

1.

$x =$	0	1	3	6	10	14	20
$y =$	3.2	2.7	1.7	0.2	− 1.8	− 3.8	− 6.8

2.

$r =$	0.5	1	1.5	2	2.5	3	3.5
$s =$	0.45	1.15	1.85	2.55	3.25	3.95	4.65

3. If p represents the pull in pounds required to lift a weight w, in an experiment with a differential pulley, find the formula connecting p and w.

$w =$	10	20	30	40	50	60	100
$p =$	3.4	4.9	6.4	7.9	9.4	10.9	16.9

4. The length of a copper wire, when it was stretched by attaching varying weights (w), showed the following elongations (e). Find the approximate formula connecting e and w.

$w =$	10	20	30	40	50	60	70
$e =$	0.05	0.10	0.14	0.21	0.26	0.31	0.35

5. From this table, find the formula connecting l and d:

$l =$	5	8	10	14	19	20	25
$d =$	21	33.6	42	58.8	79.8	84	105

6. Varying weights (w) were attached to a spiral spring. The length (l) of the spring was read for each weight, as shown in the table. Find the formula connecting w and l.

$w =$	3	5	6	7	8	9	10
$l =$	4.9	5.5	5.8	6.1	6.4	6.7	7

38. Solving Sets of Equations containing Three Unknowns. The methods used to solve pairs of linear equations, which were described in the preceding section, will now be extended to the case of sets of equations of the first degree in *three* unknowns.

Illustrative Example. Solve the set of equations

$$(1) \quad x + 3y + 2z = 4,$$
$$(2) \quad 3x - y - z = 8,$$
$$(3) \quad 5x - 2y + 4z = 25.$$

Solution. The terms containing y may be eliminated, first from equations (1) and (2), and then from equations (2) and (3), as follows:

(4)	$9x - 3y - 3z = 24$	Multiplying (2) by 3.
	$x + 3y + 2z = 4$	Given (1).
(5)	$10x \quad - \quad z = 28$	Adding (4) and (1).
(6)	$6x - 2y - 2z = 16$	Multiplying (2) by 2.
	$5x - 2y + 4z = 25$	Given (3).
(7)	$x \quad - 6z = -9$	Subtracting (3) from (6).

Thus we have the following pair of equations containing the same two unknowns:

$$(5) \quad 10x - z = 28, \qquad (7) \quad x - 6z = -9.$$

Solving these equations, we obtain $x = 3$ and $z = 2$.

Substituting these values for x and z in one of the original equations, and solving the resulting equation, we find that $y = -1$.

Hence the required solution is $x = 3$, $y = -1$, $z = 2$.

Check. When $x = 3$, $y = -1$, $z = 2$,

(1) L.S.$= 3 - 3 + 4 = 4$; R.S.$= 4$. | (2) L.S.$= 9 + 1 - 2 = 8$; R.S.$= 8$. | (3) L.S.$= 15 + 2 + 8 = 25$; R.S.$= 25$.

Hence the solution $x = 3$, $y = -1$, $z = 2$ satisfies the equations.

This method may be summarized as follows:

1. *Eliminate one of the unknown quantities by addition or subtraction from any pair of the given equations.*
2. *Eliminate the same unknown quantity by addition or subtraction from any other pair of the given equations.*
3. *Solve the resulting pair of equations and thus find the values of the two unknown quantities which they contain.*
4. *Substitute these values in one of the given equations, and thus obtain the value of the third unknown quantity.*
5. *Check the results by substitution in each of the given equations.*

NOTE. The elimination of one of the three unknown quantities in a set of equations may also be accomplished by substitution. We may solve any one of the given equations for the value of *one* unknown quantity in terms of the other *two*. The resulting expression may then be substituted in the other two equations. This produces a pair of equations containing only two unknown quantities, which may then be solved in the usual manner.

Exercises. *Solving Sets of Equations containing Three Unknowns*

Solve each of the following sets of equations. Check each set of results.

1. $x + y - z = 3,$
   $x + y + z = 7,$
   $3x - 2y + z = 2.$

2. $2a + b - 3c = 1,$
   $a - b + 2c = -4,$
   $a + 3b - c = 8.$

3. $p - q - r = 0,$
   $2p + 3q + 2r = 7,$
   $4p + 3q - 2r = -1.$

4. $2r + 3s + 4t = 3,$
   $5r + 2s = 1,$
   $s - 5t = 8.$

5. $6x + 6y - z = 0,$
   $3x = 2y,$
   $4y + 3z = 17.$

6. $\frac{1}{4}u + \frac{1}{2}v + 2w = 4,$
   $2v + 3w = -3,$
   $\frac{1}{3}u + \frac{1}{4}v - w = -2\frac{1}{2}.$

4. Problems

39. Solving Problems by Simultaneous Equations. When a problem contains several unknown quantities, it is often convenient to use two or more letters to represent them. In that case the solution of the problem requires the formation and the solution of a set of simultaneous equations, the number of equations corresponding to the number of letters. This method is to be followed in solving the following group of problems.

Exercises. *Problems involving Simultaneous Equations*

Number-Relation Problems

Solve problems 1 to 4 first by using two unknowns and then by an equation in one unknown.

1. The sum of two numbers is 95, and their difference is 21. Find the numbers.

Suggestion. The conditions of the problem may be represented by the equations $x + y = 95$ and $x - y = 21$, where x is the larger number and y is the smaller number.

2. The larger of two numbers is 4 more than three times the smaller, and their sum is 112. Find the numbers.

3. Find two numbers such that their sum is 59 and three times the smaller exceeds the larger by 13.

4. Separate 137 into two parts such that the larger is 7 less than five times the smaller.

5. The sum of two numbers is s, and their difference is d. Find the numbers.

6. Separate a number n into two parts such that the larger part is d less than twice the smaller.

Motion Problems

7. A boat covers a distance of 10 mi. in an hour while going upstream and 18 mi. in an hour while going downstream. Find the rate of the boat in still water and the rate of the current.

Suggestion. Let x (miles per hour) = the rate of the boat and y = the rate of the current. Then $x + y$ = the rate of the boat going downstream, and $x - y$ = the rate of the boat upstream.

8. Two boys rowed a boat 16 mi. downstream in 2 hr. It took 8 hr. to make the return trip. Find their rate of rowing in still water and also the rate of the current.

9. Two boys' camps were located 28 mi. apart on a river. At one camp the boys had a motor boat. They found that it took 4 hr. 40 min. to go upstream to the other camp, but that the return trip required only 2 hr. Find the rate of the motor boat and that of the current.

10. An airmail pilot made his usual trip of 165 mi. in $1\frac{1}{2}$ hr. on a certain day while going with the wind. There was no change in the velocity or direction of the wind on his return trip and it took him 1 hr. 40 min. to return. Find the speed of the airplane and the velocity of the wind.

11. It required h hours for an airplane to cover a distance of m miles while flying with the wind, and n hours while flying against the wind. Find the speed of the airplane and the velocity of the wind.

Interest Problems

12. A part of \$9500 is invested at 5% and the remainder at 6%. The annual income from both investments is \$535. How much money is invested at each rate?

13. Mr. Chambers has invested a part of \$7500 in $4\frac{1}{2}\%$ bonds and the remainder in 5% stock. The income from the bonds is \$24 more than the income from the stock. How much money has he invested at each rate?

14. A man wants to invest \$12,000, a part in 7% preferred stock and the remainder in 4% bonds, so as to average 5% on his money. How many dollars should he invest at each rate?

15. A man wishes to invest d dollars, a part at 5% and the rest at $7\frac{1}{2}\%$, so as to receive 6% on his total investment. How much money must he invest at each rate?

16. Mr. Lane has d dollars to invest. He wishes to invest part at $r\%$ and the remainder at $n\%$, so as to obtain an annual income of i dollars from both investments. How much money must he invest at each rate?

Miscellaneous Problems

17. A motion-picture theatre on a certain day sold 465 tickets, the total receipts being \$165. If each adult paid 40¢ and each child paid 25¢, how many adults were admitted? how many children?

18. The cost of sending a telegram of 12 words from one city to another is 35¢, while a telegram of 18 words costs 50¢, the charge in each case being a flat rate for the first 10 words and a uniform charge for each additional word. What is the charge for the first 10 words? for each additional word?

19. How many pounds of 60-cent tea and of 75-cent tea must be combined to make a mixture of 20 lb. to sell at 66¢ a pound?

40. Summary. In this chapter you have learned

(1) How to draw the graph of a linear function.

(2) How to solve pairs of linear equations graphically.

(3) How to solve sets of equations containing two or three unknowns algebraically.

(4) How to obtain formulas from certain tables.

(5) How to solve problems by the method of simultaneous equations.

Test on Chapter II

Solve each of the following pairs of equations graphically, and check the result:

1. $x + 4y = 24,$
   $3x - 2y = 2.$

2. $3x - 4y = 20,$
   $y = \frac{1}{6}x + 2.$

Solve each of the following sets of equations algebraically:

3. $3x - 4y = 8,$
   $6y - x = 9.$

4. $x + 2y + 3z = 2,$
   $3x + y - 4z = 3,$
   $2x - 3y = 12.$

5. Represent graphically the related values given in the table below, and from the graph obtain a formula which shows the relation of the letters used.

$p =$	3	9	21	27	33
$q =$	7	12	22	27	32

6. A temperature which gives a reading of $C°$ on a centigrade thermometer scale gives a reading of $F°$ on a Fahrenheit scale, where $F = \frac{9}{5}C + 32$. Plot F against C. From your graph estimate the temperature which gives the same readings on both scales. Find the same information algebraically and check

In the case of each of the following problems write the equations which you would use to solve it:

7. A boy starts to walk to a village 20 mi. away at the rate of 4 mi. an hour. After walking for a time, he is picked up by a passing motorist and taken to the village at the rate of 28 mi. an hour. If he reaches his destination 2 hr. after he started, how far did he walk?

8. A purse contains 14 coins consisting of nickels, dimes, and quarters. The value of the nickels and dimes is \$0.80, while the value of all the coins is \$1.55. Find the number of coins of each kind.

9. At a certain entertainment the price of admission was 35¢ for adults and 20¢ for children. If 245 persons were present and the receipts from admissions were \$73, how many adults and how many children attended?

10. A dealer plans to make up a 5-pound box of candy to sell for \$4.25, at a profit of 25% on the cost. He wants to use two kinds of candy, which cost him 60¢ and 80¢ per pound respectively. How many pounds of each kind must he use?

Exercises. *Review*

1. Multiply $2a + a^4 - 3a^2 + 1$ by $a^3 - 2 - 2a$.

2. Divide $1 - 3n^4 - n + 5n^3$ by $3n^2 + 1 - 2n$.

3. Simplify $3(a^2+b^2) - [(a^2+2ab+b^2) - (2ab-a^2-b^2)]$.

4. Find whether $a = 2$ satisfies the following equation: $3(3a - 2) + 5(5a - 6) - 4(4a - 5) - 2a - 16 = 0$.

5. Using the formula $t = \frac{2s + a}{2a}$, find the value of t when $a = 5.65$ and $s = 73.45$.

6. What value of x makes $5x + 7$ equal to $3x - 5$?

7. What value must n have in order that the expression $21c - 3c^3 + 2c^2 + c^4 + 3n$ may be exactly divisible by $c - 3 + c^2$?

8. If a dealer gains n cents on the cost of a pair of shoes, and the shoes cost d dollars, what per cent does he gain?

9. A motorist travels h hours at the rate of r miles an hour and n hours at the rate of m miles an hour. What was his average rate per hour during the entire trip?

10. If the dividend is D, the divisor is d, and the remainder is r, what is the quotient?

11. Alice is y years old now. Write the equation showing that five times her age 3 yr. ago equals three times her age 3 yr. from now.

12. If n oranges cost c cents, how many dollars will b oranges cost?

13. If $x = 3$, what is the difference in value between $-6x^2$ and $(-6x)^2$?

The invention of variables was, perhaps, the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being. Ordinary algebra simply carries to a higher stage of usefulness in a special field the device which common language employs over the whole range of discourse.—Sir T. Percy Nunn

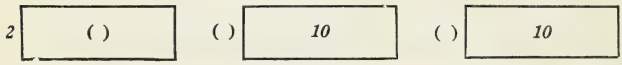
CHAPTER III

SPECIAL PRODUCTS AND FACTORING

41. Special Products. Certain combinations of algebraic expressions occur so frequently in multiplication that it is advantageous to be able to find the product in each case by means of a short method. Such cases are usually referred to as *special products.*

For example, in cases such as the following it is desirable to find the product quickly by special rules: $(x + 3)(2x - 5)$, $(3n + 1)(3n - 1)$, $(2a + b)^2$, and $(c - d)^2$.

42. Factoring. These diagrams readily suggest the fundamental relation between multiplication, division, and factoring.



In *multiplication,* the factors are given and the product is to be found. In *division,* the product and one factor are given and the other factor is to be found. In *factoring,* the product is given and the factors are to be found. In other words, *factoring is the reverse of multiplication.*

Factoring an expression is the process of finding two or more expressions whose product is the given expression.

The ability to factor an algebraic expression will be found useful in many problems involving the solution of equations or the transformation of algebraic expressions.

43. Product of a Polynomial and a Monomial, and the Factoring of Such a Product. This case is based on the Rule of Distribution for Multiplication. The relationships involved may readily be shown as follows:

MULTIPLICATION	FACTORING
$m(b + c) = mb + mc.$	$mb + mc = m(b + c).$

That is, a product may be such that each of its terms contains the same factor. The presence of such a common factor can be discovered by an inspection of all the terms of the product.

Illustrative Example 1. Factor $7\,n^4 - 21\,n^3 + 14\,n^2$.

Solution. It is evident that the common factor might be 7, n, n^2, $7\,n$, or $7\,n^2$. The *largest common* factor, namely, $7\,n^2$, is the one ordinarily used. Hence,

$$7\,n^4 - 21\,n^3 + 14\,n^2 = 7\,n^2(n^2 - 3\,n + 2).$$

NOTE The common factor may be either a monomial or a polynomial. In the latter case, the common polynomial may be treated as though it were a monomial.

Illustrative Example 2. Factor

$$2\,x(a + b) - 5\,y(a + b) + (a + b).$$

Solution. The common factor is $(a + b)$. Hence we have

$$2\,x(a + b) - 5\,y(a + b) + 1(a + b) = (a + b)(2\,x - 5\,y + 1).$$

State in words the rule which you would use in factoring polynomials whose terms have a common factor.

Checking. It is obvious that all examples in factoring may be checked either by numerical substitution or by actual multiplication of the factors. Remember, however, that neither of these methods shows whether the given expression has been factored *completely* (see page 99).

Exercises. *Common Monomial and Binomial Factors*

Numbers 1 to 10, oral

In Exercises 1 to 10, the factors are given in Column A, and the corresponding products in Column B. Cover Column B with a piece of paper and read the products for the factors given; then reverse the process by covering Column A and factoring the products which appear in Column B.

Column A	Column B
1. $3(m+n-2)$	$=3m+3n-6.$
2. $5(x+2y-3z)$	$=5x+10y-15z.$
3. $n(a-b+4)$	$=an-bn+4n.$
4. $x(2r+5s-t)$	$=2rx+5sx-tx.$
5. $2a(3a-5)$	$=6a^2-10a.$
6. $3d(3d^2-4d+1)$	$=9d^3-12d^2+3d.$
7. $xy(3x+2y)$	$=3x^2y+2xy^2.$
8. $5a^{2n}b(4-a^nb-3b^2)$	$=20a^{2n}b-5a^{3n}b^2-15a^{2n}b^3.$
9. $2\pi r^2(2r+h)$	$=4\pi r^3+2\pi r^2h.$
10. $\frac{1}{3}x(x^2+2xy-y^2)$	$=\frac{1}{3}x^3+\frac{2}{3}x^2y-\frac{1}{3}xy^2.$

Factor each of the following, and check your results:

11. $9r^2s+6rs^2-13s^3.$

12. $-3ac^4+5bc^3-c^2.$

13. $x^{3n}+4x^{2n}-6x^n.$

14. $(a-b)^2-3(a-b).$

15. $5(r-s)+(r-s)(x-y).$

16. $-2hk^3-8h^2k^2-4h^3k.$

17. $21a^3b^2+28a^2b^2-7a^2b^3.$

18. $34x^3y^3-51x^2y^4z+85xy^5z^2.$

19. $8x^2(a-b)+10x(a-b)^2.$

20. $(a+b)x+(a+b)y+(a+b)z.$

21. $3a^{n+4}-9a^{n+3}+12a^{n+2}.$

22. $1.4x^{3a}y^b-2.1x^{2a}y^{2b}-0.7x^ay^{3b}.$

44. Inserting Parentheses. It is sometimes necessary to inclose certain terms of an algebraic expression in parentheses. This may be done by removing the common factor $+1$ or -1 from the terms which are to be grouped.

Thus, if the last two terms of the expression $a - b + c$ are to be inclosed in parentheses, we may write

$$a - b + c = a + 1(-b + c) = a + (-b + c),$$

or $$a - b + c = a - 1(b - c) = a - (b - c).$$

Again, $$ax + ay - x - y = +1(ax + ay) - 1(x + y)$$
$$= (ax + ay) - (x + y).$$

These illustrations readily suggest the following rule:

A set of terms may be inclosed in parentheses preceded by a ***plus*** *sign without changing the original signs; but when they are inclosed in parentheses preceded by a* ***minus*** *sign, the sign of each term so inclosed must be changed to the* ***opposite*** *sign.*

45. Grouping to show Common Polynomial Factors. By grouping the terms of an algebraic expression it is sometimes possible to discover a common polynomial factor, thus enabling us to factor the given expression.

Illustrative Example 1. Factor $nx - ny - 2x + 2y$.

Solution. 1. $nx - ny - 2x + 2y = (nx - ny) - (2x - 2y)$
$= n(x - y) - 2(x - y)$.

2. Since $x - y$ is a common polynomial factor, we have

$$nx - ny - 2x + 2y = (x - y)(n - 2).$$

Illustrative Example 2. Factor $2ax - 3ay - 2bx + 3by + 3cy - 2cx$.

Solution. The given expression may be written in the form

$$(2ax - 2bx - 2cx) - (3ay - 3by - 3cy).$$

(Complete the solution.)

Exercises. *Inserting Parentheses and Factoring by Grouping*

Inclose within parentheses preceded by a plus sign the terms containing x or y, and inclose within parentheses preceded by a minus sign all the other terms:

1. $x - 2y - a + 3b$.
2. $3m + 12n - 9x - 36y$.
3. $9 + 4x^2 - x + 16h - k$.
4. $x^2 - 4c^2 + 2xy - 4d^2 + y^2$.
5. $ax - bm - ay + bn$.
6. $-7p + q - 14x + 21y - 9r$.
7. $x^4 - a^4 - 2a^2 - 1 - y^4$.
8. $9y^2 - 1 + 2a - a^2 - 16x^2$.
9. $x^2 - 2xy - a^2 - b^2 + y^2 - 2ab$.
10. $4cd - c^2 + x^4 - 2x^2y^2 - 4d^2 + y^4$.
11. $2mn + x^2 - 6xy - m^2 - n^2 + 9y^2$.

Factor each of the following expressions:

12. $ax - ay + bx - by$.
13. $30 - 6p - pq + 5q$.
14. $3ck - 6cl + 2dl - dk$.
15. $4x^3 + 7x^2y - 21y - 12x$.
16. $2ax - 4bx - 3ay + 6by + 5az - 10bz$.
17. $2ax - 4bx + 6cx - 3ay + 6by - 9cy$.
18. $n(a - b) - 3(b - a)$.

SUGGESTION. $-3(b - a)$ may be written $+3(a - b)$. (Why?)

19. $4m(x - y) + 5n(y - x) - 6(x - y)$.
20. $17(m - n) - 10a(m - n) + 9b(n - m)$.

Solve each of the following equations for x:

21. $3ax - 6a = 9a^2$.
22. $5cx + 5cd = 25c^2$.
23. $(m - n)x = am - an$.
24. $ax + bx = ab + b^2$.
25. $2mx - 3n^2x + 15m^2n^2 = 10m^3$.
26. $3x - 2rx = 5(3 - 2r) + (3 - 2r)(h - k)$.

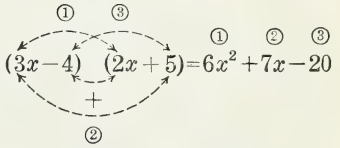
46. Finding the Product of Two Binomials. In finding the product of two binomials, we shall consider only the three cases which occur most frequently.

1. *The Product of any Two Binomials having Similar Terms.* Compare the usual method of multiplying $(3x - 4)(2x + 5)$ with the short method shown below.

LONG METHOD

$$\begin{array}{r} 3x - 4 \\ 2x + 5 \\ \hline 6x^2 - 8x \\ \quad + 15x - 20 \\ \hline 6x^2 + 7x - 20 \end{array}$$

SHORT METHOD (FOIL)



The small numbers in the circles indicate how the corresponding terms of the product are obtained. The terms connected by dotted lines are multiplied together.

Hence we infer the following method of finding the product of any two binomials having similar terms:

1. *The first term is the product of the* ***first terms*** *of the binomials.*
2. *The second term is found by combining the product of the two* ***outer terms*** *and that of the two* ***inner terms.***
3. *The third term is the product of the* ***last terms*** *of the binomials.*

NOTE. The above method may be remembered by the word ***FOIL,*** suggested by the first letters of the key words, *first, outer, inner, last.*

2. *The Square of a Binomial.* The square of any binomial may be found by writing the indicated square as a product of two equal factors and applying the above method.

Thus, $(a+b)^2=(a+b)(a+b)=a^2+2ab+b^2$.

Similarly, $(a-b)^2=(a-b)(a-b)=a^2-2ab+b^2$.

In each of the above cases you will observe that the product of the outer terms is the same as the product of the inner terms. Hence the square of a binomial may be found as follows:

1. *Square the first term.*
2. *Add twice the algebraic product of the two terms.*
3. *Add the square of the last term.*

Thus, $(2x+5)^2=4x^2+20x+25$.

Also, $[(a+2b)-3c]^2=(a+2b)^2-6c(a+2b)+9c^2$
$=a^2+4ab+4b^2-6ac-12bc+9c^2$.

Again, $1.04^2=(1+0.04)^2=1^2+2\times 0.04+0.04^2$
$=1+0.08+0.0016=1.0816$.

3. *The Product of the Sum and the Difference of Two Terms.* When two binomials having the forms $a+b$ and $a-b$ are multiplied by the FOIL method stated above, the middle term of the product will always disappear. (Why?) That is,

$$(a+b)(a-b)=a^2-b^2.$$

Hence we infer the following rule:

The product of the sum and the difference of two terms equals the square of the first term used in each of the binomials minus the square of the second term.

For example, $(5x+3)(5x-3)=25x^2-9$.

Also, $[(n-2)-4d][(n-2)+4d]=(n-2)^2-16d^2$
$=n^2-4n+4-16d^2$.

Exercises. ***Finding the Product of Two Binomials***

Numbers 1 to 20, oral

Find the following indicated products:

1. $(n+4)(n+6)$.
2. $(2a+3)(2a+1)$.
3. $(x-5)(3x+4)$.
4. $(a+3)(a-3)$.
5. $(2b-5)(2b+5)$.
6. $(7r-11)(7r+11)$.
7. $(p-q)(p-q)$.
8. $(a+3)^2$.
9. $(2x-5)^2$.
10. $(6r-5s)(8r-3s)$.
11. $(4p-3q)^2$.
12. $(9y^a+1)(9y^a-1)$.
13. $(2x^2+y)(2x^2-y)$.
14. $(3a^2+2b)^2$.
15. $(x^a-3)^2$.
16. $(12h-11k^2)(11k^2+12h)$.
17. $(-7r-10s)^2$.
18. $(2ab+3c)(ab-4c)$.
19. $(2c^2+7d)(6c^2-21d)$.
20. $(5m-9n)(-5m+9n)$.
21. $(4r^2s-5t^2)(3r^2s+7t^2)$.
22. $(5ab-2xy)(3ab-4xy)$.
23. $(6a^x-7y)(7y-6a^x)$.
24. $(\frac{1}{2}r+8s)(\frac{1}{4}r+6s)$.
25. $(\frac{2}{3}m-\frac{3}{4}n)(\frac{2}{3}m+\frac{3}{4}n)$.
26. $(\frac{3}{5}a-15)^2$.
27. $(\frac{3}{4}x^b+6y)^2$.
28. $(0.5h-6k)(0.4h+7k)$.
29. $(3a^2+0.02b)(0.6a^2-5b)$.
30. $(0.8m-0.05n^2)^2$.
31. $7(3a-4)(5a+2)$.
32. $6(4x-5y)(4x-5y)$.
33. $5a(4m-3n)^2$.
34. $(x+2)(x-2)(x^2+4)$.
35. $(30+2)(30-2)$.
36. $(90+1)(90-1)$.
37. $(50+2)^2$.
38. $(60-1)^2$.

39. 43×37.
40. 84×76.
41. 62^2.
42. 103^2.
43. $(5\frac{1}{2})^2$.
44. 1.02^2.
45. 98^2.
46. 997^2.

47. $[(a-b)-3][(a-b)+3]$.

48. $[2n+(x+y)][2n-(x+y)]$.

49. $[(m+n)-4p][(m+n)+5p]$.

50. $[(a-b)+6c]^2$.

51. $[4k+3(r+s)][k-5(r+s)]$.

52. $[7h+2(m-n)][7h-2(m-n)]$.

53. $[9(x-y)-11z^2][9(x-y)+11z^2]$.

54. $[12x^2-5(c+d)]^2$.

55. If the side s of a square is increased by 3, how much is its area increased? If each side is diminished by 4, how much is its area reduced?

56. The radius of a circular flower bed is r feet. By spading a strip 2 ft. wide around the bed, how much is its area increased?

57. In t seconds a freely falling body starting from rest drops a distance of $16t^2$ feet. How far will it drop in n seconds? How much farther will it drop in 3 more seconds?

Solve each of the following equations and check each root:

58. $(x+3)(x-2)=x^2-1$.

59. $(x+1)(x+2)+18=(x+3)(x+4)$.

60. $x^2-(x+5)^2=8x-7$.

61. $21-(2y+3)(2y-3)=2y(5-2y)$.

62. $(5r+4)(5r-4)-8r=(5r-3)^2+2r+5$.

63. $6x^2-(2x-1)^2=(2x+7)(x+8)$.

64. $(2n+1)(8n-5)-(4n+3)(4n-3)=0$.

65. $(3x+2)^2-(3x-1)^2+15=9x$.

47. Factoring a Trinomial having the Form $ax^2 + bx + c$. In factoring trinomials having the form $ax^2 + bx + c$, we shall find it convenient to consider two cases which occur rather frequently.

1. *General Case.* When two binomials having similar terms, such as $4x + 3$ and $2x - 1$, are multiplied together, the product is a trinomial having the form $ax^2 + bx + c$. Hence we can obtain the factors of such a trinomial by reversing the method explained in the preceding section.

Illustrative Example. Factor $5x^2 - 6x - 8$.

Solution. The binomial factors must satisfy three requirements: (1) *the first terms must be factors of* $5x^2$; (2) *the last terms must be factors of* -8; (3) *these factors must be chosen in such a manner that when the product of the outer terms is combined with the product of the inner terms, the result is* $-6x$.

Hence we must select the required binomials from possible combinations such as appear in the following list:

	Trial Factors	Check for Middle Term
(1)	$(5x + 8)(x - 1)$	$-5x + 8x = +3x$
(2)	$(5x + 1)(x - 8)$	$-40x + x = -39x$
(3)	$(5x - 1)(x + 8)$	$40x - x = 39x$
(4)	$(5x - 2)(x + 4)$	$20x - 2x = 18x$
(5)	$(5x + 4)(x - 2)$	$-10x + 4x = -6x$

Only the last of the above combinations meets all three requirements. Hence $5x^2 - 6x - 8 = (5x + 4)(x - 2)$.

The number of trials may be considerably reduced by remembering that when the last term of the given trinomial is *positive*, its factors must have *like signs*; and when it is *negative*, its factors must have *unlike signs*.

2. *Special Case. The Trinomial Square.* When a trinomial has the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$, it is sometimes called a *trinomial square*, or a "perfect trinomial square."

In such a case the two factors of the first term are alike and the two factors of the last term are alike. Hence the two resulting binomial factors are alike. That is, the given trinomial may be written as the square of a binomial.

Thus, $x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$.

Also, $4x^2 + 12xy + 9y^2 = (2x + 3y)(2x + 3y) = (2x + 3y)^2$.

Exercises. *Factoring Trinomials*

Numbers 1 to 10, oral

Factor each of the following trinomials:

1. $n^2 + 4n + 3$.

2. $a^2 + 4a + 4$.

3. $b^2 - 3b - 4$.

4. $y^2 - 6y + 9$.

5. $n^2 + 3n - 10$.

6. $8 - 2c - c^2$.

7. $16 - 8k + k^2$.

8. $5a^2 + 1 + 6a$.

9. $6s + 9s^2 + 1$.

10. $12y^2 - 11y + 2$.

11. $1 + 5x + 6x^2$.

12. $4t^2 + 12t + 9$.

13. $11r^2 + 21r - 2$.

14. $m^2 + 2mn + n^2$.

15. $2a^2 + 5ab + 2b^2$.

16. $3c^2 - 3d^2 + 8cd$.

17. $9r^2 + s^2 - 6rs$.

18. $42y + 49 + 9y^2$.

19. $6a^2 - 13ab + 6b^2$.

20. $9c^2 - 30cd + 25d^2$.

21. $49x^2 + 28xy + 4y^2$.

22. $20r^2 - r - 30$.

23. $32m^2 + 18mn - 45n^2$.

24. $4h^4 - 4h^2k + k^2$.

25. $10r^2 - 21s^4 + rs^2$.

26. $2v^6 - 5v^3 - 12$.

27. $12w^3 + 9 + 4w^6$.

28. $a^{2n} - 12a^n + 36$.

29. $3x^{2a} + 4x^a - 7$.

30. $25x^{4a} + 20x^{2a}y^b + 4y^{2b}$.

31. $0.09r^2 + 2.4r + 16$.

32. $3.2n^2 - 0.2n - 7.5$.

33. $\frac{3}{4}y^2 + 4y + 4$.

34. $\frac{1}{4}x^2 - 3xy + 9y^2$.

35. $(m-n)^2+5(m-n)+6$.

36. $(r+s)^2+8(r+s)+16$.

37. $3(a-b)^2-11(a-b)+8$.

38. $10(c+d)^2-7(c+d)-12$.

39. $6a^2+7a(x-y)-5(x-y)^2$.

40. $25k^2-20k(m+2n)+4(m+2n)^2$.

41. $(a+b)^2-4(a+b)(c-d)+4(c-d)^2$.

The first column of the following table gives the areas of certain rectangles or squares. Find the missing items.

	Area	Dimensions	Perimeter
42.	$3w^2+w-2$	$(3w-2)(w+1)$	$8w-2$
43.	$s^2+10s+25$		
44.	$2l^2-7l-30$	$(2l+5)(l-6)$	$6l-2$
45.	$12b^2-23b-39$		
46.	$9e^2-42e+49$	$(3e-7)(3-7)$	$12e-28$,

Solve the following equations for x:

47. $(a+b)x=a^2+2ab+b^2$.

48. $(3n-2)x=3n^2-14n+8$.

49. $rx+2x=r^2+4r+4$.

50. $3ax-x=9a^2-6a+1$.

51. $4nx-5x=20n^2-17n-10$.

52. $7bx+6=2x+19b+7b^2$.

48. Factoring Completely. The process of factoring a given algebraic expression should, in general, be continued until each factor contains no other factors besides itself and

unity. The resulting factors are called the *simplest factors* of the original expression.

Thus, $18x^4 + 30x^3 - 12x^2 = 6x^2(3x^2 + 5x - 2)$
$= 3 \cdot 2 \cdot x \cdot x(3x - 1)(x + 2).$

However, we shall understand "factoring completely" to mean that only *polynomial* factors are to be refactored whenever that is possible.

Thus, the expression given above is said to be "factored completely" when it is written in the form

$$18x^4 + 30x^3 - 12x^2 = 6x^2(3x - 1)(x + 2).$$

In all the exercises which follow, unless there are directions to the contrary, the expressions to be factored are to be resolved into their simplest factors, or to be factored completely, in the sense explained above.

Exercises. *Factoring Completely*

Factor completely:

1. $3x^2 - 12x + 12$.
2. $4r^2 - 16rs + 12s^2$.
3. $3a^2b^2 + 13a^2b + 12a^2$.
4. $2c^3 - 12c^2d + 18cd^2$.
5. $8c^3 - 24c^2 + 16c$.
6. $5ax - 5ay + 10bx - 10by$.
7. $m^3n - 3m^3 - 2m^2n + 6m^2$.
8. $14r^3s - 35r^2s + 14rs$.
9. $5x^2y + 10xy^2 + 5y^3$.
10. $6p^2q^2 - 12pq^3 - 18q^4$.
11. $4k^2 + 16k^3 + 12k^4$.
12. $-18a^2b + 24ab^2 - 8b^3$.
13. $6r^2s^2 - 36r^2s^3 + 6r^2s$.
14. $32c^3 - 16c^2 + 2c$.
15. $\frac{1}{3}x^2y + \frac{2}{3}xy^2 + \frac{1}{3}y^3$.
16. $n^4 + 2n^2 - n^3 - 2n$.
17. $2m^2h - 4m^2k + 8mh - 16mk$.
18. $20a^{3n} - 2a^{2n} - 6a^n$.
19. $3y^{n+2} - 3y^{n+1} - 18y^n$.

20. $4\,a^{2x+1}b^{2y} + 8\,a^{x+1}b^{2y} + 4\,ab^{2y}$.

21. $x^2(a+b) - 3\,x(a+b) - 4(a+b)$.

22. $4(r-s) - 12\,r(r-s) + 9\,r^2(r-s)$.

23. $3(m-n)^2 + 12(m-n) + 9$.

24. $9\,a(c-d) - 24\,a(c-d)^2 + 16\,a(c-d)^3$.

49. Factoring the Difference of Two Squares. From the fact that $(a+b)(a-b) = a^2 - b^2$, it follows that the factors of $a^2 - b^2$ are $a+b$ and $a-b$. That is,

$$(a^2 - b^2) = (a+b)(a-b).$$

Hence we have the following rule:

The difference of two squares may be factored by finding the square root of each of the squares in the given order and using as factors the sum and the difference of these roots.

NOTE. Remember that the "square root" of a given expression is a quantity which when multiplied by itself produces the given expression. Thus, the square root of 9 is 3, the square root of $25\,a^2b^4$ is $5\,ab^2$, and so on.

Illustrative Example 1. Factor $a^2 - 9\,b^2$.

Solution. $a^2 - 9\,b^2 = (a+3\,b)(a-3\,b)$.

Illustrative Example 2. Factor $25\,x^2 - (2\,y-3)^2$.

Solution. Since the square root of $25\,x^2$ is $5\,x$, and the square root of $(2\,y-3)^2$ is $2\,y-3$, we have

$$\begin{aligned} 25\,x^2 - (2\,y-3)^2 &= [5\,x + (2\,y-3)][5\,x - (2\,y-3)] \\ &= (5\,x + 2\,y - 3)(5\,x - 2\,y + 3). \end{aligned}$$

Illustrative Example 3. Factor $a^2 - 2\,ab + b^2 - 16\,c^2$.

Solution. The first three terms form a trinomial square and hence may be written in the form $(a-b)^2$. That is,

$$\begin{aligned} a^2 - 2\,ab + b^2 - 16\,c^2 &= (a^2 - 2\,ab + b^2) - 16\,c^2 = (a-b)^2 - 16\,c^2 \\ &= [(a-b) + 4\,c][(a-b) - 4\,c] \\ &= (a - b + 4\,c)(a - b - 4\,c). \end{aligned}$$

Illustrative Example 4. Factor $9n^2 - 4a^2 - 4a - 1$.

Solution. If the last three terms are grouped together and the common factor -1 is removed, we obtain a trinomial square. Thus,

$$\begin{aligned} 9n^2 - 4a^2 - 4a - 1 &= 9n^2 - 1(4a^2 + 4a + 1) = 9n^2 - (2a + 1)^2 \\ &= [3n + (2a + 1)][3n - (2a + 1)] \\ &= (3n + 2a + 1)(3n - 2a - 1). \end{aligned}$$

Exercises. *Factoring the Difference of Two Squares*

Numbers 1 to 15, oral

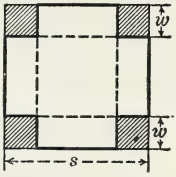
Factor each of the following expressions:

1. $n^2 - 16$.
2. $9x^2 - 25$.
3. $49 - 4y^2$.
4. $64a^2 - b^2$.
5. $36c^2 - 25d^2$.
6. $0.16x^2 - 49y^2$.
7. $100 - 0.09n^4$.
8. $\frac{1}{4}k^2 - m^6n^2$.
9. $x^2y^4 - \frac{4}{9}$.
10. $a^{2n} - b^{2n}$.
11. $a^4 - b^2$.
12. $c^4 - 25$.
13. $1 - a^{2x}$.
14. $81 - x^6y^{10}$.
15. $k^2 - 625$.

Factor completely each of the following:

16. $2y^3 - 18y$.
17. $a^2b^5 - a^4b^3$.
18. $63s^2t - 28t^3$.
19. $80d^5 - 5d$.
20. $ax^{3n} - a^3x^n$.
21. $8a^{2n+1} - 50a$.
22. $(a + b)^2 - x^2$.
23. $4(m - n)^2 - 1$.
24. $k^2 - (c + d)^2$.
25. $m^2 + 2mn + n^2 - w^2$.
26. $4a^2 - 4ab + b^2 - 9$.
27. $c^2 - 6cd - 16 + 9d^2$.
28. $x^2 - r^2 - 2rs - s^2$.
29. $4h^2 - 4k^2 + 4h + 1$.
30. $12w - 9 - 4w^2 + 16r^2$.
31. $20xy - 25y^2 + 1 - 4x^2$.
32. $(a - b)^2 - (x + y)^2$.
33. $9(c + d)^2 - 4(c - d)^2$.
34. $m^2 - 2mn + n^2 - (p^2 + 2pq + q^2)$.
35. $4a^2 - 4ab + b^2 - c^2 - 6cd - 9d^2$.
36. $25x^2 + 60xy + 36y^2 + 14z - z^2 - 49$.
37. $a^2 - 4b^2 + c^2 - 9d^2 - 2ac + 12bd$.

38. From the corners of a square sheet of metal four equal squares have been cut and the sides folded up to form an open box. The dimensions are shown in the figure.



(1) Write a formula for the area (A) of the inside surface of the box.

(2) Rewrite in factored form the formula obtained in (1).

(3) Find the value of A if $s = 32$ in. and $w = 4$ in., using (a) the formula in (1), and (b) the formula in (2). Which of the two processes is more convenient?

39. A conduit pipe contains 484 telephone wires. The inside diameter of the pipe is $2R$, and the diameter of each wire is $2r$.

(1) Write a formula for the area (A) of the unoccupied space in a cross section of the conduit.

(2) Rewrite in factored form the formula obtained in (1).



Royal Canadian Air Force

Peace Tower of the Parliament Buildings at Ottawa

In planning and erecting modern buildings, the extensive use of mathematical formulas is necessary

50. Factoring the Sum or the Difference of Two Cubes. If we divide the expression $a^3 + b^3$ by $a + b$, we shall obtain the quotient $a^2 - ab + b^2$. Likewise, when $a^3 - b^3$ is divided by $a - b$, the quotient is $a^2 + ab + b^2$.

Hence, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Observe that the sign connecting the terms of the *binomial* factor is the *same* as that appearing in the original binomial, but that the sign of the middle term of the *trinomial* factor is the *opposite* of that in the binomial.

Illustrative Example 1. Factor $c^3 + 27\,d^3$.

Solution. $c^3 + 27\,d^3 = (c + 3\,d)(c^2 - 3\,cd + 9\,d^2)$.

Illustrative Example 2. Factor $(x + 3)^3 - y^3$.

Solution. $(x + 3)^3 - y^3 = [(x + 3) - y][(x + 3)^2 + y(x + 3) + y^2]$

$= (x + 3 - y)(x^2 + 6\,x + 9 + xy + 3\,y + y^2)$.

Exercises. *Factoring the Sum or the Difference of Two Cubes*

Factor each of the following expressions:

1. $x^3 + y^3$.
2. $a^3 + 8$.
3. $27 + d^3$.
4. $r^3 - s^3$.
5. $8 - n^3$.
6. $a^3b^3 - 64$.
7. $x^3 + 0.027\,y^3$.
8. $0.008\,t^3 - 1$.
9. $0.125 + c^3d^3$.
10. $\frac{1}{8} + 64\,n^3$.
11. $8\,x^3 - 27\,y^3$.
12. $a^3b^6 + c^3$.
13. $r^3s^9 - 216$.
14. $x^6 + y^6$.
15. $a^6 - b^9$.
16. $a^{3n} + b^{3n}$.
17. $x^{3a} - y^{3b}$.
18. $x^{3n+3} - y^{6n}$.

Factor completely each of the following:

19. $x^4 + x$.
20. $3\,n^3 - 24$.
21. $54\,m^2 + 2\,m^5n^3$.
22. $c^4d - cd^7$.
23. $a^{3n+2}b^n + a^2b^{n+3}$.
24. $a^6 - b^6$.

SUGGESTION. In Exercise 24 regard the expression first as the difference of two squares.

25. $64 - r^6$.
26. $r^8s^2 - r^2s^8$.
27. $d^6 - 7\,d^3 - 8$.
28. $a^4 + a^3b + 27\,a + 27\,b$.

51. The Factor Theorem. There are many expressions the factoring of which lies beyond the scope of the methods considered thus far. Some of these expressions can be factored very readily by the application of a simple principle that may be inferred from a comparison of results such as the following:

1. If $2x^2 - 5x + 7$ is divided by $x - 3$, there will be a remainder of 10.

2. The value of $2x^2 - 5x + 7$ when $x = 3$ is also 10.

In general, *the value of a polynomial in x,* when n is substituted for x, is the same as the remainder obtained when the polynomial is divided by* $x - n$.

Proof. 1. Let $P(x)$ represent any polynomial in x, which is to be divided by $x - n$.

Denote the quotient by Q and the remainder by R.

2. Then $P(x) = (x - n) \cdot Q + R$. (Why?)

3. If n is substituted for x in this equality, we have

$$P(n) = (n - n) \cdot Q + R.$$

4. That is, $$P(n) = R.$$

NOTE. The abbreviation $P(n)$ means the value of the polynomial $P(x)$ when $x = n$.

Suppose, now, that the value of n is such that when $x = n$ the value of the given polynomial is zero. In that case $P(n) = 0$. Hence, in the proof above, we find that

$$P(n) = 0 = R.$$

That is, in such a case the remainder (R) equals zero, and therefore $x - n$ is an exact divisor, or a factor, of $P(x)$.

* The term "polynomial in x" is here understood to refer to an expression of the form $ax^k + bx^{k-1} + \cdots + px + q$, in which k is a positive integer and the coefficients are constants; for example, $6x^3 - \frac{1}{2}x^2 + 7x - 3$.

We thus have the following *Factor Theorem*:

If any polynomial in x becomes equal to 0 when n is substituted for x, then $x - n$ is a factor of the polynomial.

Illustrative Example 1. Factor $2x^3 - 7x^2 + 12$.

Solution. Any binomial of the form $x - n$ which is to be a factor of $2x^3 - 7x^2 + 12$ must be such that its second term is a factor of $+12$. (Why?) Hence, in this case, n might have any of the following values: 1, 2, 3, 4, 6, 12, or the corresponding negative values.

If $x = 1$, then $2x^3 - 7x^2 + 12 = 2 - 7 + 12 = 7$, which is not zero. Hence $x - 1$ is not a factor of $2x^3 - 7x^2 + 12$.

If $x = 2$, then $2x^3 - 7x^2 + 12 = 16 - 28 + 12 = 0$. Hence $x - 2$ is a factor of the given expression. By division we find that the other factor is $2x^2 - 3x - 6$, and therefore

$$2x^3 - 7x^2 + 12 = (x - 2)(2x^2 - 3x - 6).$$

Illustrative Example 2. Factor $x^5 + 32y^5$.

Solution. If $x = -2y$, then $x^5 + 32y^5 = -32y^5 + 32y^5 = 0$. Hence $x - (-2y)$, or $x + 2y$, is a factor of the given expression. The other factor is obtained by division, and so we have

$$x^5 + 32y^5 = (x + 2y)(x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4).$$

Exercises. *Using the Factor Theorem*

1. Evaluate $x^3 + 4x - 5$ when $x = 1$. Is $x - 1$ a factor of $x^3 + 4x - 5$?

2. Evaluate $x^3 - 6x + 4$ when $x = 2$. Is $x - 2$ a factor of $x^3 - 6x + 4$?

3. Evaluate $r^3 + 4r^2 - 8$ when $r = -2$. Is $r + 2$ a factor of $r^3 + 4r^2 - 8$?

4. Show that $x - y$ is a factor of $x^3 + xy^2 - 2y^3$.

5. Show that $x - 3$ is a factor of $x^3 - 2x^2 - 9$.

6. Show that $a + 3b$ is a factor of $a^5 + 243b^5$.

Factor each of the following expressions:

7. $x^3 - x^2 - 4$.
8. $y^3 + 2y + 3$.
9. $r^3 - 10r - 3$.
10. $x^3 - 7x + 6$.
11. $s^3 - 19s - 30$.
12. $a^3 + 5a^2 + 2a - 8$.
13. $d^3 + 5d^2 + 7d + 3$.
14. $2n^3 - 5n^2 - n - 6$.
15. $x^4 - 4x^3 - x^2 + 16x - 12$.
16. $y^4 - 2y^3 - 7y^2 + 8y + 12$.
17. $x^5 - 32$.
18. $n^5 + 243$.
19. $y^7 + 128$.
20. $a^5 - b^5$.
21. $m^5 - 1024n^5$.
22. $a^5 + b^5c^5$.
23. $c^5 - 32d^{10}$.
24. $r^3 - 7rs^2 - 6s^3$.
25. $2x^3 + 7x^2y + 4xy^2 + 3y^3$.
26. $a^3 - 8a^2b + 17ab^2 - 10b^3$.

Exercises. *Review*

Factor completely each of the following expressions:

1. $64x^2 - 121$.
2. $16a^3 - 250$.
3. $27b^2 - 48b^4$.
4. $49n^2 - 42n + 9$.
5. $8 + 3r - 26r^2$.
6. $28x^2 - 3x - 18$.
7. $16r^3 + 56r^2s + 49rs^2$.
8. $144ab^2 - 169a^3$.
9. $81m^2 + 24m^5$.
10. $4 - 108d^6$.
11. $98xy^3 - 450x^5y$.
12. $75x + 240x^2 + 192x^3$.
13. $60y^2 - 72y + 21$.
14. $10n^3 + 116n^2 + 66n$.
15. $64s^2 - 48st + 9t^2$.
16. $64a^6 - 1$.
17. $x^3 - 12x + 16$.
18. $2y^3 - 3y^2 + 2y - 3$.
19. $2a^2 - 4ab + 2b^2 - 8$.
20. $2c^3 + 3c^2 - 18c + 8$.
21. $400 - 169k^4$.
22. $2n + 64n^6$.
23. $32x^4 + 108x$.
24. $18a^4 + 50a^2 - 60a^3$.

25. $21\,cd - 54\,c^2 + 20\,d^2$.

26. $4\,a^{4n} - 9\,x^{2n}$.

27. $x^{n+3} + 8\,x^n y^6$.

28. $16\,x^{2a+2} - x^2 y^{4b}$.

29. $d^{5n+1} - 243\,d$.

30. $16\,a^{3x} + 81\,a^x - 72\,a^{2x}$.

31. $14\,n^{s+2} - 70\,n^s - 29\,n^{s+1}$.

32. $2\,r^3 + r^2 - 8\,r - 4$.

33. $2\,mn - 15 + 5\,n - 6\,m$.

34. $4\,x^2 - y^2 - 9 + 6\,y$.

35. $16\,a^2 - 36\,b^2 + 1 + 64\,a^4$.

36. $6\,s^3 - 25\,s^2 + 2\,s + 8$.

37. $0.06\,c^2 - 0.7\,cd - 20\,d^2$.

38. $12\,x^3 - 0.75\,x$.

39. $0.027 - k^6$.

40. $2\,d + 1.6\,d^2 + 0.32\,d^3$.

41. $(a+b)^2 + 13(a+b) + 36$.

42. $36\,a^3 - a(2\,x - y)^2$.

43. $x^{2a} - y^{2b}$.

44. $169 - 25\,c^{4n}$.

45. $r^{4c} - s^{2d}$.

46. $x^{2k} - h^{2p}$.

47. $a^{2n} + 2\,a^n b^n + b^{2n}$.

48. $x^{2a} - 7\,x^a - 60$.

49. $6\,a^{2x} + 7\,a^x - 20$.

50. $s^{2m} + 3\,s^m - 10$.

51. $4\,k^4 - 4\,k^2(m + 2\,n) + (m + 2\,n)^2$.

52. $9\,a(3\,c + d) - a^3(3\,c + d)$.

53. $2(a+b)^2 + 7(a+b)(x-y) + 6(x-y)^2$.

52. Solving Equations by Factoring. Certain types of equations can readily be solved by factoring, with the aid of the following fundamental principle:

If the product of two or more factors is zero, at least one of the factors must be equal to zero.

Thus, let it be required to solve the equation

$$x^2 - 5\,x - 14 = 0.$$

It is evident that this equation may be written in the form

$$(x - 7)(x + 2) = 0.$$

Then it follows from the principle stated above that either

$$(1)\ x - 7 = 0 \quad \text{or} \quad (2)\ x + 2 = 0.$$

Hence $$x = 7 \quad \text{or} \quad x = -2.$$

We can readily verify by substitution the fact that *either* of these results will satisfy the given equation, thus showing that the given equation has *two* roots, namely, 7 and -2.

An equation in one unknown which contains the *square* of the unknown quantity but no higher power, such as $3x^2 + 10x - 12 = 0$, is said to be of the *second degree.* Such an equation is usually called a *quadratic equation.* You will find that *every quadratic equation has two roots.* Similarly, an equation of the *third degree,* such as the equation $x^3 - x^2 - 4x + 4 = 0$, has *three* roots, and so on.

To solve equations in one unknown, of the second degree or a higher degree, by factoring, apply the following rule:

1. *Transform the given equation, if necessary, so that the right member is zero.*
2. *Factor the left member completely and set each factor which contains the unknown equal to zero.*
3. *Solve the resulting equations.*
4. *Check each solution.*

In the case of applied problems, negative solutions must be rejected whenever they appear to have no meaning.

Exercises. *Solving Equations by Factoring*

Solve each of the following equations, and check the roots:

1. $n^2 - 8n + 7 = 0.$
2. $x^2 - x - 12 = 0.$
3. $d^2 - 16 = 0.$
4. $a^2 + 6a + 9 = 0.$
5. $y^2 + 7y + 10 = 0.$
6. $2k^2 + 5k - 3 = 0.$
7. $4r^2 - 4r + 1 = 0.$
8. $4n^2 - 25 = 0.$

9. $x^2 - 6 = 5x$.

10. $s^2 - 6s = 16$.

11. $r^2 - 9r = 0$.

*12. $6x^2 = 21x$.

13. $c^2 - 0.81 = 0$.

14. $d^2 - 1.1d + 0.3 = 0$.

15. $4x^2 + 0.64 = 3.2x$.

16. $n^2 - \frac{4}{9} = 0$.

17. $y^2 + \frac{1}{2}y - 7\frac{1}{2} = 0$.

18. $a^2 + 2\frac{1}{4} = 3a$.

19. $t^2 - 2\frac{1}{4}t + \frac{1}{2} = 0$.

20. $c^3 - 36c = 0$.

21. $n^3 + 5n^2 - 14n = 0$.

22. $2x^3 + 4x^2 = 30x$.

23. $4y^3 = 81y$.

24. $a^3 - a^2 - 4a + 4 = 0$.

25. $c^3 + 3c^2 = c + 3$.

26. $d^3 - 6d^2 + 11d - 6 = 0$.

27. $x^4 - 7x^2 - 6x = 0$.

28. $a^4 - 17a^2 + 16 = 0$.

29. $r^2 - 18r + 81 = 0$.

30. $c^4 - 13c^2 + 36 = 0$.

31. $y(2y + 15) = 27$.

32. $(t - 7)(t + 3) = -9$.

33. $a(a + 2) - 18 = a + 2$.

34. $(2x-7)(x-4)=3(x-4)$.

35. $(n-2)^2-6(n-2)+9=0$.

36. $k(k^2 - 9) = 4(k^2 - 9)$.

37. A rectangle whose length is 6 in. more than its width contains 160 sq. in. Find its dimensions.

38. The altitude of a triangle is 3 ft. shorter than the base. The area is 44 sq. ft. Find the base and the altitude.

39. A rectangular flower bed whose dimensions are 10 ft. and 15 ft. is to be doubled in area by increasing each of its dimensions by the same amount. How many feet must be added to each dimension?

* In exercises of this type, it is not permissible to divide by the unknown. For in this case the resulting equation would be $6x = 21$, thus giving but one root, namely, $x = 3\frac{1}{2}$. Instead, it is necessary to transform the equation by writing $6x^2 - 21x = 0$, from which we have $3x(2x - 7) = 0$. Hence the two roots which satisfy the given equation are 0 and $3\frac{1}{2}$.

53. Summary. In this chapter you have learned

(1) How to inclose a set of terms in parentheses.

(2) How to find the following special products:

- (*a*) The product of a polynomial and a monomial.
- (*b*) The product of any two binomials.
- (*c*) The square of any binomial.
- (*d*) The product of the sum and the difference of two terms.

(3) How to deal with the following cases in factoring:

- (*a*) Removing a common monomial or polynomial factor.
- (*b*) Factoring by grouping to show a common polynomial factor.
- (*c*) Factoring a trinomial of the form $ax^2 + bx + c$.
- (*d*) Factoring a trinomial square.
- (*e*) Factoring the difference of two squares.
- (*f*) Factoring the sum or the difference of two cubes.
- (*g*) Factoring by means of the Factor Theorem.

(4) How to use special products and factoring in solving equations and problems.

Test on Chapter III

Find each of the following indicated products:

1. $(2a + 3)(4a - 5)$.
2. $(5r - s^2)(5r + s^2)$.
3. $(6a - c^2d)^2$.
4. $(x^n - 4y)(3x^n + 7y)$.

Factor completely each of the following expressions:

5. $20r^2 - 5r^4$.
6. $8 + 27d^3$.
7. $16x^2 - 24x + 9$.
8. $2y^3 + y^2 - 3y$.
9. $3(s - 2t) + r(2t - s)$.
10. $(c + 3d)^2 - 25$.
11. $18n^2 + 12n + 2$.
12. $n - 64n^4$.
13. $2y^2 - 0.7y + 0.03$.
14. $a^3 + 2a^2b - ab - 2b^2$.
15. $a^2 - b^2 + 4b - 4$.
16. $x^4 - x^2 - 12$.
17. $a^{2n} - 2a^nb^2 + b^4$.
18. $2x^{2a+1} + 4x^{a+1} - 16x$.

19. Evaluate $x^3 - 3x^2 + 4$ when $x = 2$. Is $x - 2$ a factor of $x^3 - 3x^2 + 4$?

20. Find a binomial factor of $y^3 - 7y - 6$.

21. Find the trinomial factor of $a^3 + 2a - 3$.

22. Factor $n^5 + 32$.

23. Solve the equation $(x - 3)(5x + 4) = (2x - 3)^2 - 9$.

24. Solve the equation $2x^3 + 3x^2 = 9x$.

25. Four times the square of a certain whole number is equal to nine times the next consecutive number. Find the number.

26. The perimeter of a rectangle is 56 ft. Its area is 192 sq. ft. Find the dimensions of the rectangle.

Exercises. *Review*

1. Multiply $x^{2n} - 2x^n + 4$ by $x^n + 2$, and check your result by division.

2. Simplify:
$(a - 1)(a - 2) - 3a(a + 3) + 2[(a + 2)(a + 1) - 3]$.

3. Solve for x and y:
$$2x + y = d + 1,$$
$$x = 2y + d - 1.$$

4. Inclose the last three terms of $4ab - 3x + 5a - 12$ in parentheses preceded (1) by a plus sign; (2) by a minus sign.

5. Given the equation $y = \frac{3}{4}x + 9$. State (1) the slope of the graph of this equation; (2) the y-intercept.

6. Given the formula $r = \frac{Re}{E - e}$. Find the value of r to the nearest tenth when $E = 7.6$, $e = 2.3$, and $R = 52.7$.

7. How will $x - \frac{1}{x}$ change as x increases from 1 to 10?

8. The product of two factors is n. One of the factors is a. What is the other factor?

9. Express a length of f feet and i inches in inches.

10. The cost of planting 100 young apple trees is d dollars. How many can be planted for s dollars?

In the case of each of the following statements indicate whether it is true or false:

11. The cube of the difference of two numbers equals the difference of their cubes.

12. If the side of a square is doubled, its area is doubled.

13. The difference between two negative numbers may be a positive number.

14. The product of two binomials may be a binomial.

Mathematics is often considered a difficult and mysterious science, because of the numerous symbols which it employs. Of course, nothing is more incomprehensible than a symbolism which we do not understand. Also, a symbolism which we only partially understand and are unaccustomed to use is difficult to follow. In exactly the same way the technical terms of any profession or trade are incomprehensible to those who have never been trained to use them. But this is not because they are difficult in themselves. On the contrary they have invariably been introduced to make things easy. So in mathematics, granted that we are giving any serious attention to mathematical ideas, the symbolism is invariably an immense simplification.—A. N. WHITEHEAD

CHAPTER IV

FRACTIONS

54. Algebraic Fractions. The fractional form is frequently used in algebra, as it is in arithmetic, to express the *indicated quotient* of two numbers.

Evidently the numerator or the denominator of a fraction, or both, may contain an algebraic expression. A fraction of this type is an *algebraic fraction.*

Thus, $\frac{n}{2}$, $\frac{3}{a}$, $\frac{d}{t}$, $\frac{x+2}{3x}$, $\frac{5}{3n-4}$, and $\frac{x^2+5x+6}{x+3}$ are algebraic fractions.

55. The Fundamental Principle of Fractions. When fractions are to be simplified or combined by any one of the four fundamental operations, it is often necessary to *change their form without changing their value.* To do this, we shall find it necessary to make use of the following

Fundamental Principle

The numerator and the denominator of a fraction may be multiplied by the same expression, or divided by the same expression, without changing the value of the fraction.

For example, $\frac{1}{4}=\frac{1\times 3}{4\times 3}=\frac{3}{12}$, and $\frac{10}{16}=\frac{10\div 2}{16\div 2}=\frac{5}{8}$.

Similarly, $\frac{a}{b}=\frac{a\cdot m}{b\cdot m}=\frac{am}{bm}$, and $\frac{an}{bn}=\frac{an\div n}{bn\div n}=\frac{a}{b}$.

Two fractions which have the same value but not the same form are said to be equivalent.

NOTE. In dealing with fractions, care should be exercised to avoid transformations that are not based on the Fundamental Principle.

For example, $\frac{3}{4}$ does *not* equal $\frac{3+1}{4+1}$, or $\frac{4}{5}$;

$\frac{5}{6}$ does *not* equal $\frac{5-3}{6-3}$, or $\frac{2}{3}$;

$\frac{2}{3}$ does *not* equal $\frac{2^2}{3^2}$, or $\frac{4}{9}$;

$\frac{9}{25}$ does *not* equal $\frac{\sqrt{9}}{\sqrt{25}}$, or $\frac{3}{5}$.

Observe, in particular, that *we cannot remove, or cancel, like terms from the numerator and the denominator of a fraction. Only like factors can be removed, or cancelled.*

Thus, $\frac{a+b}{a+c}$ is *not* the same as $\frac{\not{a}+b}{\not{a}+c}$, or $\frac{b}{c}$.

Again, $\frac{3a-4b}{5ac}$ does *not* equal $\frac{3\not{a}-4b}{5\not{a}c}$, or $\frac{3-4b}{5c}$,

since *a is not a factor of the entire numerator, but only of one term of the numerator.*

Oral Exercises. *Applying the Fundamental Principle*

In each of the following exercises explain how the second fraction was obtained from the first:

1. $\frac{5}{6}=\frac{10a}{12a}$.

2. $\frac{21}{28}=\frac{3}{4}$.

3. $\frac{7n}{9}=\frac{28n^2}{36n}$.

4. $\frac{4x^3}{-8xy}=\frac{x^2}{-2y}$.

5. $\frac{c}{-d}=\frac{-c}{d}$.

6. $\frac{2a+4b}{8}=\frac{a+2b}{4}$.

7. $\frac{7a^2}{2x-5y}=\frac{21a^2x}{6x^2-15xy}$.

8. $\frac{2a}{a-b}=\frac{2a(a+b)}{a^2-b^2}$.

Change each of the following fractions to an equivalent fraction having the denominator indicated:

9. $\frac{5\,d}{6}$; 30.

10. $\frac{2\,n}{3\,x}$; $6\,x^3$.

11. $\frac{4\,a^2}{a-b}$; $2\,b-2\,a$.

12. $\frac{5\,a}{d+3}$; $d^2+6\,d+9$.

13. $\frac{4(r+s)}{r^2-s^2}$; $r-s$.

14. $\frac{k-2}{k+4}$; $(k+4)(k-2)$.

Supply the missing term in each of the following exercises:

15. $\frac{35\,x^3}{5\,x(2\,x+y)}=\frac{(\quad)}{2\,x+y}$.

16. $\frac{4\,ab}{-9\,cd}=\frac{-12\,a^2b}{(\quad)}$.

17. $\frac{-17\,n}{y-x}=\frac{(\quad)}{x-y}$.

18. $\frac{10}{3\,x}=\frac{(\quad)}{6\,x(k+5)}$.

19. $\frac{a+2\,b}{7\,a^2+14\,ab}=\frac{1}{(\quad)}$.

20. $\frac{n+3}{2\,n+1}=\frac{(\quad)}{(n-3)(2\,n+1)}$.

56. Reducing Fractions to Lowest Terms. A fraction is said to be in its lowest terms when the numerator and the denominator have no common factor except 1. In order to change a fraction to such a form, we must apply the *Fundamental Principle of Fractions* (page 114). That is, in this case, we see that

To reduce a fraction to its lowest terms, we must divide the numerator and the denominator by the product of all the common factors.

Illustrative Example 1. Reduce to lowest terms the fraction $\frac{6\,a^3b}{15\,a^2b^3}$.

Solution.
$$\frac{6\,a^3b}{15\,a^2b^3}=\frac{\overset{1}{\cancel{2}}\cdot\overset{a}{\cancel{3}}\cdot\overset{1}{\cancel{a^2}}\cdot\cancel{b}}{\underset{1}{\cancel{3}}\cdot 5\cdot\underset{1}{\cancel{a^2}}\cdot\underset{b^2}{\cancel{b^3}}}=\frac{2\,a}{5\,b^2}.$$

Illustrative Example 2. Reduce to lowest terms the fraction $\frac{2x^2 - 5x - 12}{4x + 6}$.

Solution. $\frac{2x^2 - 5x - 12}{4x + 6} = \frac{\overset{1}{\cancel{(2x+3)}}(x-4)}{\underset{1}{\cancel{2(2x+3)}}} = \frac{x-4}{2}$.

NOTE. Remember that when we divide any expression by itself, the result is 1. Thus, $\frac{a}{a} = 1$, $\frac{a+b}{a+b} = 1$, and so on.

57. Changing Signs in a Fraction. A fraction may also be transformed by changing its signs according to the method explained below. Since a fraction is an indicated quotient, three signs must be considered: the sign of the numerator (or the dividend), the sign of the denominator (or the divisor), and the sign before the fraction (or the sign of the quotient).

Thus, by applying the rule of signs for division, we readily obtain equivalent quotients such as the following:

$$+\frac{+6}{+2} = +\frac{-6}{-2} = -\frac{+6}{-2} = -\frac{-6}{+2} = +3.$$

Similarly, $+\frac{+a}{+b} = +\frac{-a}{-b} = -\frac{+a}{-b} = -\frac{-a}{+b}$.

From such illustrations we infer the following principle:

The value of a fraction is not altered if we change (1) *the sign of the numerator and the sign of the denominator, or* (2) *the sign before the fraction and also the sign of the numerator or the denominator.*

Illustrative Example. Reduce to lowest terms the fraction $\frac{3x - 3y}{y^2 - x^2}$.

Solution. $\frac{3x - 3y}{y^2 - x^2} = -\frac{3(x-y)}{x^2 - y^2} = -\frac{\overset{1}{3\cancel{(x-y)}}}{(x+y)\underset{1}{\cancel{(x-y)}}} = -\frac{3}{x+y}$.

NOTE. Changing the sign of a *polynomial* numerator or denominator involves changing the sign of *each term* in the polynomial.

Observe, moreover, that we may multiply or divide both the numerator and the denominator of a fraction by -1 without changing the value of the fraction. Hence changes of sign such as those mentioned under (1) above may also be explained by means of the *Fundamental Principle of Fractions* (see page 114).

For example, $\dfrac{2}{3-x}=\dfrac{2(-1)}{(3-x)(-1)}=\dfrac{-2}{x-3}$.

Exercises. *Reduction of Fractions*

Reduce each of the following fractions to its lowest terms:

1. $\dfrac{15\,ab}{35\,ac}$.
2. $\dfrac{-40\,xy}{72\,x^2}$.
3. $\dfrac{78\,c^3d}{-143\,cd^2}$.
4. $\dfrac{9\,rs}{108\,r^2s}$.
5. $\dfrac{-92\,a^2b^2}{-23\,b^2}$.
6. $\dfrac{16\,a(x+y)}{28\,b(x+y)}$.
7. $\dfrac{2\,c^2(m-n)}{3\,cd(m-n)}$.
8. $\dfrac{-14(a+b)^2}{56(a+b)}$.
9. $\dfrac{5\,m(r-s)^2}{n(r-s)^3}$.
10. $\dfrac{18(x-y)}{24(y-x)}$.
11. $\dfrac{-5\,abc}{ab+ac}$.
12. $\dfrac{16\,x^3}{4\,x^2-8\,xy}$.
13. $\dfrac{ac-bc}{c^2+cd}$.
14. $\dfrac{3\,k^2+12\,k}{15\,k-6\,k^2}$.
15. $\dfrac{mr+ms}{3\,nr+3\,ns}$.
16. $\dfrac{28-4\,w}{21\,l-3\,lw}$.
17. $\dfrac{2\,ab-bx}{3\,x-6\,a}$.
18. $\dfrac{c^2-d^2}{ac+ad}$.
19. $\dfrac{a^2+ab}{a^2+2\,ab+b^2}$.
20. $\dfrac{c^2-3\,c+2}{c^2-4}$.
21. $\dfrac{2\,x^2-3\,xy-2\,y^2}{4\,x^2-9\,xy+2\,y^2}$.
22. $\dfrac{5\,r^2-rs-4\,s^2}{r^3-rs^2}$.
23. $\dfrac{x^3-7\,x+6}{x^2+x-6}$.
24. $\dfrac{8\,c^3-27}{8\,c^2-6\,c-9}$.

Which of the following statements are correct? In those which are incorrect point out the error.

25. $\frac{3\,a}{6\,b} = \frac{a}{2\,b}$.

26. $\frac{4\,xy}{2\,x+5} = \frac{2\,y}{5}$.

27. $\frac{5\,rst}{rs-rt} = \frac{5\,st}{s-t}$.

28. $\frac{a^2-ab}{a^2-b^2} = \frac{a}{b}$.

29. $\frac{a^2-ab}{a^2-b^2} = \frac{a}{a+b}$.

30. $\frac{x-y}{y+x} = -1$.

31. Observe that $\frac{19}{95} = \frac{1}{5}$. A pupil wrote $\frac{1\not{9}}{\not{9}5} = \frac{1}{5}$. Was his method correct?

32. A pupil wrote $\frac{a+b}{a^2+2\,ab+b^2} = \frac{a+b}{(a+b)^2} = a+b$. Explain the error made.

33. Indicate which of the following fractions it is possible to reduce to lower terms:

(1) $\frac{16\,a^2}{36\,b^2}$. (2) $\frac{(a-5)^2}{(b+2)^2}$. (3) $\frac{c+d}{c^2}$. (4) $\frac{cd}{c(a+b)}$. (5) $\frac{x^2-y^2}{(x-y)^2}$. (6) $\frac{4\,m}{m+7\,n}$.

58. Multiplication of Fractions. The multiplication of algebraic fractions is performed, as in arithmetic, in accordance with the following rule:

To find the product of two or more fractions, write the product of the numerators over the product of the denominators, and reduce the resulting fraction to its lowest terms.

Illustrative Example. Multiply $\frac{x^2-2\,x-8}{x^2-9}$ by $\frac{2\,x-6}{x+2}$.

Solution. In examples of this type, the work may be shortened by at once expressing the numerator and the denominator of each fraction in factored form. That is,

$$\frac{x^2-2\,x-8}{x^2-9}\cdot\frac{2\,x-6}{x+2} = \frac{(x-4)\cancel{(x+2)}\cdot 2\cancel{(x-3)}}{(x+3)\cancel{(x-3)}\cdot\cancel{(x+2)}} = \frac{2(x-4)}{x+3}.$$

(1 over the first cancelled factor, 1 over the second cancelled factor)

Check. The work may be checked by numerical substitution. Care should be taken, however, that no value of x be chosen which would cause any denominator to become zero, since division by zero is always to be excluded (see page 34).

59. Division of Fractions. To divide one algebraic fraction by another, we proceed exactly as in the case of arithmetical fractions. That is, we apply the following rule:

Invert the divisor and proceed as in multiplication.

Illustrative Example. Divide $\frac{6a^2}{a^2-4b^2}$ by $\frac{3a^2-6ab}{a+2b}$.

Solution.

$$\frac{6a^2}{a^2-4b^2} \div \frac{3a^2-6ab}{a+2b} = \frac{\overset{2\,a}{\cancel{6a^2}} \cdot \overset{1}{\cancel{(a+2b)}}}{\underset{1}{\cancel{(a+2b)}}(a-2b) \cdot \underset{1}{\cancel{3a}}(a-2b)} = \frac{2a}{(a-2b)^2}.$$

Check. The work may be checked by numerical substitution.

NOTE. The fraction $\frac{x}{y}$ is said to be the *reciprocal* of $\frac{y}{x}$, and conversely. Hence the rule stated above may be expressed as follows: *To divide an expression by a fraction, we may multiply the expression by the reciprocal of the fraction.*

Exercises. *Multiplication and Division of Fractions*

Perform the indicated operations:

1. $\frac{15}{28} \times \frac{4}{5}$.
2. $\frac{27}{32} \div \frac{9}{20}$.
3. $35 \div \frac{10}{11}$.
4. $\frac{9a}{16b} \cdot \frac{2}{3}$.
5. $\frac{6}{5x} \cdot \frac{x^2}{2y}$.
6. $\frac{5c}{28d^2} \cdot 21cd$.
7. $-8m^2 \cdot \frac{5n}{18m}$.
8. $\frac{14a^2}{15b} \div \frac{4}{5}$.
9. $\frac{8c}{21d^2} \div \frac{6c^2}{7d}$.
10. $\frac{12xy}{25z} \div 8x^2$.
11. $\left(\frac{3a^2}{10b}\right)^2 \cdot \frac{25b^2}{36a}$.
12. $10x \cdot \frac{x-7}{5}$.
13. $\frac{-8k}{5} \cdot \frac{15-10k}{4}$.
14. $\frac{2m+3}{6} \div \frac{2m}{9}$.
15. $\frac{6}{7a} \div \frac{3b}{a^2-ab}$.

16. $\dfrac{m+n}{4x-6y} \cdot \dfrac{12}{m+n}$.

17. $\dfrac{18\,a^3}{a-b} \cdot \dfrac{b-a}{3\,a^2-6\,ab}$.

18. $\dfrac{6\,mn}{k^2+4} \cdot \dfrac{k^4-16}{3\,k-6}$.

19. $\dfrac{5\,x^2+10\,xy}{4\,r^2-36\,s^2} \cdot \dfrac{2\,r-6\,s}{x^2-4\,y^2}$.

20. $\dfrac{4-x^2}{m^2-n^2} \div \dfrac{2-x}{m+n}$.

21. $\dfrac{8\,c^2-18\,d^2}{5\,a^2+b^2} \div (6\,ac+9\,ad)$.

22. $\dfrac{8\,ab}{5\,y-15} \div \dfrac{16\,ac}{30-10\,y}$.

23. $\dfrac{s^3-t^3}{s^3+t^3} \div \dfrac{3\,s-3\,t}{rs+rt}$.

24. $\dfrac{m^2-6\,m+9}{3\,m-6} \cdot \dfrac{6\,m^2-12\,m}{4\,m-12}$.

25. $\dfrac{a^{2x}-b^{2y}}{a^{x+1}+4\,a^x} \cdot \dfrac{2\,ab+8\,b}{2\,a^{2x}-a^xb^y-b^{2y}}$.

26. $\dfrac{k^3+9\,k}{k^2} \div \dfrac{k^4-81}{k^2-6\,k+9}$.

27. $\dfrac{r^2-4\,rs+4\,s^2}{4\,rs+10\,s} \cdot \dfrac{2\,r+5}{r^3-s^3} \div \dfrac{r^2-4\,s^2}{6\,rs^2-6\,s^3}$.

28. $\dfrac{2\,a^2-5\,ab-3\,b^2}{18\,a^2+3\,a-1} \cdot \dfrac{c^2-36\,a^2c^2}{2\,a^2-7\,ab+3\,b^2} \div \dfrac{2\,ac+bc}{6\,a^2+5\,a+1}$.

60. Changing Fractions to Equivalent Fractions; Lowest Common Denominator. Fractions cannot be combined by addition or subtraction unless they have the same denominator. For this reason it is often necessary to change fractions into equivalent fractions having a common denominator. This is done by applying the *Fundamental Principle of Fractions* (see page 114).

The common denominator of two or more fractions must contain each of the given denominators as an exact divisor. To avoid unnecessary work, it is customary to use the *lowest common denominator* (*L.C.D.*), as illustrated by the following examples.

Illustrative Example 1. Change $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{7}{30}$ to respectively equivalent fractions having the lowest common denominator.

Solution. $6 = 2 \cdot 3$

$8 = 2 \cdot 2 \cdot 2$, or

and $30 = 2 \cdot 3 \cdot 5$.

$2\overline{|6 \cdot 8 \cdot 30}$

$3\overline{|3 \cdot 4 \cdot 15}$

$\overline{|1 \cdot 4 \cdot 5}$

Hence the L.C.D. $= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$. L.C.D. $= 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

That is, $\frac{5}{6} = \frac{100}{120}$, $\frac{7}{8} = \frac{105}{120}$, and $\frac{7}{30} = \frac{28}{120}$.

Illustrative Example 2. Change $\frac{5x}{3x^2 - 12}$ and $\frac{3}{x^2 + 2x}$ to respectively equivalent fractions having the lowest common denominator.

Solution. $3x^2 - 12 = 3(x + 2)(x - 2)$,

and $x^2 + 2x = x(x + 2)$.

Hence the L.C.D. $= 3x(x + 2)(x - 2) = 3x(x^2 - 4)$.

That is, $\frac{5x}{3x^2 - 12} = \frac{5x}{3(x^2 - 4)} \times \frac{x}{x} = \frac{5x^2}{3x(x^2 - 4)}$.

Similarly, $\frac{3}{x^2 + 2x} = \frac{3}{x(x + 2)} \times \frac{3(x - 2)}{3(x - 2)} = \frac{9(x - 2)}{3x(x^2 - 4)}$.

Hence, to change two or more fractions to respectively equivalent fractions having the lowest common denominator, we have the following rule:

1. *Separate each denominator into its simplest factors.*
2. *Find the product of all the different factors, using each factor the greatest number of times it appears in any one denominator. This product is the L.C.D.*
3. *Multiply the numerator and the denominator of each fraction by those factors of the L.C.D. which are not factors of the denominator of that fraction.*

Exercises. ***Finding the Lowest Common Denominator***

In each case change the given fractions to respectively equivalent fractions having the lowest common denominator:

1. $\frac{4}{9}, \frac{-16}{27}$.
2. $\frac{7}{12}, \frac{-8}{15}, \frac{11}{20}$.
3. $\frac{2}{a}, \frac{3}{a^2}$.
4. $\frac{a}{x^2y}, \frac{-b}{xy^2}$.
5. $\frac{1}{rs}, \frac{2s}{rt}, \frac{3r}{st}$.
6. $\frac{4a}{a+3}, \frac{a}{5}$.
7. $\frac{m+2n}{m^2-n^2}, \frac{3}{m-n}$.
8. $\frac{6c}{c^2-d^2}, \frac{5}{d-c}$.
9. $\frac{x-2}{8x^3}, \frac{x+5}{6x}$.
10. $\frac{9a+1}{a^2+ab}, \frac{2a-3}{a+b}$.
11. $\frac{c-2}{c+5}, \frac{-12}{c+2}$.
12. $\frac{4y}{x+y}, \frac{6}{x}, \frac{3}{x-y}$.
13. $\frac{y-1}{y+3}, \frac{y+2}{y^2+4y+3}$.
14. $\frac{a+4}{a^2-6a+9}, \frac{2-a}{3-a}$.
15. $\frac{4d}{16-d^2}, \frac{1}{d^2-6d+8}$.
16. $\frac{n^2+4}{n^3-8}, \frac{n+2}{n^2+2n+4}, \frac{-2n}{2-n}$.
17. $\frac{2a-1}{a^{2n}-25}, \frac{3a}{a^{n+2}+5a^2}, \frac{4a}{a^{n+1}-5a}$.

61. Addition and Subtraction of Fractions. When two or more algebraic fractions are to be combined by addition or subtraction, the result is obtained as in arithmetic.

Illustrative Example 1. Combine $\frac{2n+3}{6n} + \frac{n-5}{4n^2} - \frac{1}{3}$.

Solution. The L.C.D. is $12n^2$.

Hence
$$\frac{2n+3}{6n} + \frac{n-5}{4n^2} - \frac{1}{3} = \frac{2n(2n+3)}{12n^2} + \frac{3(n-5)}{12n^2} - \frac{4n^2}{12n^2}$$
$$= \frac{2n(2n+3) + 3(n-5) - 4n^2}{12n^2}$$
$$= \frac{4n^2 + 6n + 3n - 15 - 4n^2}{12n^2}$$
$$= \frac{9n-15}{12n^2} = \frac{3n-5}{4n^2}.$$

Check. Let $n = 2$. Substitute this value in the given expression and in the final result.

Then $\frac{2n+3}{6n}+\frac{n-5}{4n^2}-\frac{1}{3}=\frac{4+3}{12}+\frac{2-5}{16}-\frac{1}{3}=\frac{7}{12}-\frac{3}{16}-\frac{1}{3}=\frac{1}{16}.$

Also, $\frac{3n-5}{4n^2}=\frac{6-5}{16}=\frac{1}{16}.$

Illustrative Example 2. Combine $\frac{3x}{x^2-4y^2}-\frac{x-y}{x^2+2xy}.$

Solution. The L. C. D. is $x(x+2y)(x-2y)$.

Hence

$$\frac{3x}{x^2-4y^2}-\frac{x-y}{x^2+2xy}=\frac{3x\cdot x}{x(x+2y)(x-2y)}-\frac{(x-y)(x-2y)}{x(x+2y)(x-2y)}$$

$$=\frac{3x^2-(x^2-3xy+2y^2)}{x(x+2y)(x-2y)}$$

$$=\frac{3x^2-x^2+3xy-2y^2}{x(x+2y)(x-2y)}$$

$$=\frac{2x^2+3xy-2y^2}{x(x+2y)(x-2y)}$$

$$=\frac{\cancel{(x+2y)}(2x-y)}{x\cancel{(x+2y)}(x-2y)}=\frac{2x-y}{x(x-2y)}.$$

Check. The work may be checked by numerical substitution, as in the preceding example.

NOTE. An expression which is part integral and part fractional, such as $x+3-\frac{2}{x}$, is called a *mixed expression.* Such an expression may be changed into a single fraction by transforming the integral part into a fraction having the denominator 1 and proceeding as indicated above.

Illustrative Example 3. Express as a single fraction

$$\frac{3x}{x-3}-x+5.$$

Solution.

$$\frac{3x}{x-3}-x+5=\frac{3x}{x-3}-\frac{x-5}{1}$$

$$=\frac{3x}{x-3}-\frac{(x-5)(x-3)}{x-3}$$

$$=\frac{3x-(x^2-8x+15)}{x-3}$$

$$=\frac{3x-x^2+8x-15}{x-3}$$

$$=\frac{-x^2+11x-15}{x-3}.$$

Check. The work may be checked by numerical substitution.

From the above examples we see that the following is the rule for adding and subtracting fractions:

1. *If necessary, change the given fractions to respectively equivalent fractions having their lowest common denominator.*
2. *Combine the resulting numerators, according to the sign appearing before each fraction, using parentheses when convenient, and write this result over the L.C.D.*
3. *Remove all parentheses and write the algebraic sum of the resulting terms over the L.C.D.*
4. *Reduce the resulting fraction to its lowest terms.*

Exercises. ***Addition and Subtraction of Fractions***

Perform the indicated operations:

1. $\frac{5}{6}+\frac{11}{18}-\frac{4}{9}$.
2. $3-\frac{7}{12}-\frac{13}{15}$.
3. $\frac{n}{6}+\frac{5n}{8}+\frac{3n}{4}$.
4. $2k+\frac{5k}{-12}-\frac{9k}{16}$.
5. $\frac{x^2}{5}-\frac{11x}{30}+\frac{7}{15}$.
6. $\frac{1}{d}+\frac{2}{3d}-\frac{3}{2d}$.
7. $\frac{5}{a}+\frac{4}{a^3}+\frac{7}{-a^2}$.
8. $1-\frac{5}{6s}-\frac{3}{4s^2}$.
9. $\frac{2k-5}{3}-\frac{k-2}{4}-\frac{1}{6}$.
10. $\frac{4}{9}-\frac{2-d^2}{d^2}+\frac{2}{3d}$.
11. $\frac{5-2x}{-x^2}+\frac{4x-1}{x}-4$.
12. $\frac{2m+1}{m}-3-\frac{1-2n}{2n}$.
13. $\frac{3}{y-5}+\frac{2}{y}+\frac{1}{2}$.
14. $\frac{4}{3a}-\frac{a+b}{a^2-ab}$.
15. $\frac{d}{d-3}+\frac{-3}{d+3}$.
16. $\frac{x+12}{x^2-16}-\frac{3}{x-4}$.

17. $\dfrac{10}{d^2-25}+\dfrac{1}{5-d}+\dfrac{1}{5+d}$.

18. $\dfrac{7}{2a^2+6a}+\dfrac{5-3a^2}{9-a^2}-3$.

19. $\dfrac{6}{8x^2-18y^2}+\dfrac{1}{2x^2+3xy}-\dfrac{1}{4xy-6y^2}$.

20. $\dfrac{2}{n+2}-\dfrac{n+3}{4-n^2}-\dfrac{3n+1}{n^2-4n+4}$.

21. $\dfrac{4s-t}{s^2+st-2t^2}+\dfrac{s+t}{2st-2s^2}+\dfrac{5}{2s}$.

22. $\dfrac{1}{2}-\dfrac{5x}{2x-6}-2x-6$.

23. $\dfrac{3}{a+2}+\dfrac{a+1}{a^2-9}-\dfrac{a-1}{a^2+5a+6}$.

24. $\dfrac{y+2}{y^2-3y+2}-\dfrac{y-2}{2-y-y^2}-\dfrac{1-y}{4-y^2}$.

62. Complex Fractions. Whenever a fraction contains one or more fractions either in its numerator or in its denominator, or in both, it is called a *complex fraction.*

Thus, $\dfrac{\frac{3}{5}}{4}$, $\dfrac{a+b}{\frac{c}{d}}$, and $\dfrac{n+\frac{2}{3}}{n^2-\frac{n}{3}+\frac{3}{4}}$ are complex fractions.

Complex fractions occur in many formulas. They also arise in the numerical checking of the algebraic processes and in checking fractional roots of equations.

A complex fraction may be considered as an example in division and be simplified accordingly.

Thus, $\dfrac{n+\frac{1}{2}}{\frac{3}{4}}=\left(n+\dfrac{1}{2}\right)\div\dfrac{3}{4}=\dfrac{2n+1}{2}\times\dfrac{4}{3}=\dfrac{(2n+1)\cdot\overset{2}{\cancel{4}}}{\underset{1}{\cancel{2}}\cdot 3}=\dfrac{2(2n+1)}{3}$.

Instead, it is usually advisable to apply the *Fundamental Principle of Fractions.* That is, we may *multiply both the numerator and the denominator by the L.C.D. of all the fractions appearing in the given complex fraction.* The resulting fraction should then be reduced to lowest terms.

Illustrative Example. Simplify $\dfrac{2-\dfrac{8}{a^2}}{\dfrac{a}{2}+1}$.

Solution. The L.C.D. of the denominators is $2a^2$. Hence

$$\frac{2-\frac{8}{a^2}}{\frac{a}{2}+1}=\frac{2a^2\left(2-\frac{8}{a^2}\right)}{2a^2\left(\frac{a}{2}+1\right)}=\frac{4a^2-16}{a^3+2a^2}=\frac{1}{\frac{a^2(a+2)}{4(a+2)(a-2)}}=\frac{4(a-2)}{a^2}.$$

Exercises. *Complex Fractions*

Simplify each of the following complex fractions:

1. $\dfrac{\frac{1}{8}}{1-\frac{3}{8}}$.

2. $\dfrac{6}{3-\frac{1}{3}}$.

3. $\dfrac{4-\frac{2}{5}}{4}$.

4. $\dfrac{\frac{1}{2}-\frac{1}{4}}{\frac{1}{2}+\frac{1}{3}}$.

5. $\dfrac{6-\frac{4}{7}}{1+\frac{5}{14}}$.

6. $\dfrac{\frac{3a^2}{4x^2}}{\frac{15ab}{32xy}}$.

7. $\dfrac{\frac{x}{3}+\frac{2x}{5}}{\frac{x^2}{5}}$.

8. $\dfrac{\frac{3}{4}-n}{\frac{3}{8}+n}$.

9. $\dfrac{y-\frac{1}{y}}{\frac{1}{y}+1}$.

10. $\dfrac{x-\frac{2}{3}}{3x-\frac{4}{3x}}$.

11. $\dfrac{\left(\frac{a}{b}\right)^2-9}{\frac{a}{b}+3}$.

12. $\dfrac{1-\frac{d}{c}}{c-\frac{d^3}{c^2}}$.

13. $\dfrac{\frac{x^2-y^2}{ab+3b^2}}{\frac{x-y}{a^2+3ab}}$.

14. $\dfrac{\dfrac{x}{y-z}-1}{\dfrac{x-y+z}{y^2-2yz+z^2}}$. 15. $\dfrac{1-\dfrac{d-7}{d^2-9}}{d+\dfrac{4d-6}{d-3}}$. 16. $\dfrac{\dfrac{2x^2+x-6}{3x^2+5x-2}}{\dfrac{4x^2-12x+9}{3x^2+11x-4}}$.

17. Simplify the right member of the following formulas:

(1) $i=\dfrac{E}{R+\dfrac{r}{2}}$. (2) $x=\dfrac{\dfrac{D-d}{2}}{\dfrac{l}{L}}$.

18. Does $x=\frac{2}{3}$ satisfy the equation $\dfrac{x^2}{8}+\dfrac{x+1}{4x-2}=2\frac{5}{9}$?

63. Summary. In this chapter you have learned

(1) How to apply the Fundamental Principle of Fractions.
(2) How to change the signs of a fraction.
(3) How to combine fractions by addition, by subtraction, by multiplication, and by division.
(4) How to simplify complex fractions.

Test on Chapter IV

In each of the following cases explain how the second fraction was obtained from the first:

1. $\dfrac{4a}{2a-3b}=\dfrac{12a^2}{6a^2-9ab}$.
2. $\dfrac{ax+ay}{x^2-y^2}=\dfrac{a}{x-y}$.
3. $\dfrac{c-d}{m-n}=\dfrac{d-c}{n-m}$.
4. $\dfrac{\frac{1}{2}r}{\frac{3}{4}s}=\dfrac{2r}{3s}$.

Supply the missing term in each of the following:

5. $\dfrac{3n}{n+1}=\dfrac{(\quad)}{nd+d}$.
6. $\dfrac{x-2y}{5}=\dfrac{2x^2-8y^2}{(\quad)}$.
7. $\dfrac{16t^2}{2st-4rt}=\dfrac{(\quad)}{2r-s}$.

8. Reduce to lowest terms: $\frac{4\,a^2+12\,a+9}{8\,a^3+27}$.

9. Given $\frac{a}{b}=\frac{c}{d}$. If $b=d$, does $a=c$?

10. Simplify: $\left(\frac{3}{xy^2}\right)\left(1+\frac{y^2}{x^2}\right)\div\left(\frac{2}{x^2}+\frac{2}{y^2}\right)$.

11. Combine: $\frac{-10\,d}{d^2-16}-2+\frac{7-3\,d}{4-d}$.

12. Select the word within the parentheses which will correctly complete the following statement: If the fraction $\frac{m}{n}$ is less than 1, then its reciprocal is (greater, less) than 1.

13. Determine by substitution whether or not $x=\frac{3\,a}{5}$ satisfies the equation $\frac{a}{2\,x}-\frac{x}{2\,a}=\frac{8\,x}{9\,a}$.

14. Given the equation $M=a+\frac{b}{c+\frac{d}{e}}$. If a, b, c, and d remain constant while e increases, does M increase or decrease?

15. A man travels for h hours at the rate of r miles per hour and for t hours at the rate of v miles per hour. What is his average rate for the distance covered?

Exercises. *Review*

Factor completely each of the following:

1. $x^{2n}-x^n-12$.
2. $10\,x-7\,x^2-6\,x^3$.
3. $18\,x^4-8\,y^4$.
4. $a^4-(5\,a-6)^2$.
5. $a^{6n}-b^6$.
6. $bc^2-3\,ac^2+12\,a^2-4\,ab$.

7. Given that $y = mx + b$. If $y = -1$ when $x = 1$, and $y = 5$ when $x = 4$, find the values of m and b.

8. Find the value of c if the graph of $y = \frac{2}{3}x + c$ passes through the point whose coördinates are (6, 1).

9. Solve the following set of equations, and check your solution:

$$3x - 2y + 5z = 26,$$
$$x - 2y + 3z = 6,$$
$$2x + 3y = 4z + 20.$$

10. Find the quotient and the remainder when

$$30c^4 + 12 - 5c + 11c^3 - 82c^2$$

is divided by $3c^2 - 4 + 2c$.

11. A man has d dollars and q quarters. How much money has he left after spending n quarters?

12. If n tablets of a certain kind cost c cents, how many of these tablets can be bought for d dollars?

13. Solve the equation $(n - 6)^2 - (2n - 5)^2 = 16$.

14. If $y = \frac{12 - x}{x}$, does y increase or decrease as x increases from 1 to 6?

15. Find the value of $\frac{3d^2 - 2d - 7}{2d^3 - 1}$ when $d = -\frac{3}{2}$.

16. If the list price of an article is represented by L and the discount a merchant offers from the list price is represented by $r\%$, how would you represent the selling price in terms of L and r?

Mathematics is the glory of the human mind.—LEIBNIZ

CHAPTER V

FRACTIONAL EQUATIONS; RATIO AND PROPORTION

1. Fractional Equations containing One or More Unknowns

64. Fractional Equations. An equation that contains one or more fractions in which the unknown occurs in the denominator is called a *fractional equation.* Such an equation may readily be changed into an equation that does not involve fractions by multiplying both members by an expression which contains each of the denominators among its factors. (Why?) The L. C. D. of the given fractions is the multiplier that is found most convenient for this purpose. This process of transforming a fractional equation is called *clearing the equation of fractions.* The resulting equation can then be solved in the usual way.

Equations containing fractional coefficients, such as $\frac{2}{3}x + 2x = \frac{1}{5}x + 8$ and $\frac{x}{4} + \frac{x+2}{6} = \frac{2x}{3}$, which were considered in Chapter I, will be reviewed in the present chapter, since they are also solved by the method suggested above.

Illustrative Example 1. Solve the equation

$$\frac{5x-3}{4} - \frac{x-2}{8} = \frac{2x+4}{3}.$$

Solution. The L. C. D. is 24. Multiplying each member by 24, we have

$$\frac{24 \cdot (5x-3)}{4} - \frac{24 \cdot (x-2)}{8} = \frac{24 \cdot (2x+4)}{3}.$$

Reducing each fraction to its lowest terms, we obtain

$$6(5x-3)-3(x-2)=8(2x+4).$$

Applying the Rule of Distribution, we have

$$30x-18-3x+6=16x+32.$$

Then $$27x-12=16x+32,$$

and $$x=4.$$

Check. Substitute 4 for x in each side of the original equation.

Illustrative Example 2. Solve the equation

$$\frac{g^2-7}{g^2-25}-\frac{1}{2}=\frac{g-1}{2g-10}.$$

Solution. The L.C.D. is $2(g+5)(g-5)$.

(1)	$\frac{2(g+5)(g-5)\cdot(g^2-7)}{(g+5)(g-5)}-\frac{2(g+5)(g-5)}{2}$ $=\frac{2(g+5)(g-5)\cdot(g-1)}{2(g-5)}.$	$M_{L.C.D.}$
(2)	$2(g^2-7)-(g+5)(g-5)=(g+5)(g-1).$	Why?
(3)	$2g^2-14-g^2+25=g^2+4g-5.$	Rule of Distr.
(4)	$16=4g.$	Tr.
(5)	$g=4.$	D_4.

Check. Substitute 4 for g in each side of the original equation.

Exercises. *Equations containing Fractions*

Numbers 1 to 10, oral

Solve each of the following equations, and check each solution:

1. $\frac{3x}{8}=6.$
2. $\frac{n-5}{4}=1.$
3. $10=\frac{2x-2}{3}.$
4. $\frac{2x}{3}+5=9.$
5. $3-\frac{n}{6}=\frac{2}{3}.$
6. $\frac{n}{3}+\frac{n}{5}=8.$
7. $\frac{k}{6}+\frac{1}{2}=\frac{k}{5}.$
8. $\frac{12}{d}+3=7.$
9. $\frac{3}{y}-10=\frac{1}{2y}.$
10. $\frac{1}{5}-\frac{2}{x}=\frac{1}{10}.$
11. $\frac{5}{4}+\frac{22}{w}=4.$
12. $\frac{8}{x}-\frac{10}{3x}=\frac{7}{3}.$
13. $\frac{5}{2n}=\frac{2}{3}-\frac{5}{6n}.$

14. $\frac{1}{2x}+\frac{1}{12}-\frac{2}{5x}=0.$

15. $\frac{r-13}{6}=\frac{31-5r}{4}.$

16. $\frac{c-3}{3}+\frac{1}{2}=\frac{c+2}{4}.$

17. $\frac{3d+5}{2}+\frac{5d-3}{3}=11.$

18. $\frac{2d+4}{3}-\frac{7d-13}{27}=\frac{6d+7}{9}.$

19. $\frac{x-1}{2}-\frac{7x+1}{20}=\frac{2x+1}{5}-3.$

20. $\frac{1}{2}(a+2)+\frac{2}{3}(2a-5)=23\frac{1}{3}.$

21. $0.055x-0.04(710-x)=0.1.$

22. $9x-42-\frac{0.6x-0.2}{0.05}=\frac{5x-20}{2.5}.$

23. $\frac{4}{7}(2y+1)-\left(y-\frac{y}{4}\right)=\frac{y}{4}.$

24. $\frac{x+6}{3}-2\left(1-\frac{x+2}{4}\right)=1.$

25. $\frac{d+4}{d}-3=\frac{2}{d}.$

26. $\frac{4a-3}{6a}=\frac{8a-6}{5a}.$

27. $\frac{x-3}{4x}+\frac{x-8}{x}=\frac{5}{8}.$

28. $\frac{2x+1}{2x}-\frac{5-3x}{3x}=4\frac{1}{3}.$

29. $\frac{2y}{3}-\frac{y^2-3}{2y}+\frac{1}{2}=\frac{y}{6}.$

30. $\frac{4}{x}+\frac{x-1}{5}=\frac{3}{x}+1.$

SUGGESTION. The above equation becomes a quadratic equation when cleared of fractions. It can then be solved by factoring.

31. $\frac{d^2-4}{d}-\frac{d+7}{3}=d.$

32. $\frac{3}{x+1}=\frac{2}{x+2}+\frac{1}{x-2}.$

33. $\frac{r+4}{2r+4}-\frac{r+3}{3r+6}=\frac{5}{6}.$

34. $\frac{2}{x}-\frac{x+1}{4x}=\frac{1}{x}.$

35. $\frac{4n-5}{4n}-\frac{n+2}{12n}=\frac{1}{2}-\frac{2}{3n}.$

36. $3-\frac{2n-8}{24n}=3\frac{1}{4}-\frac{3n+6}{6n}.$

37. $\frac{210}{3n-2}=\frac{50}{n}.$

38. $\frac{10-7n}{6-7n}=\frac{5n-4}{5n}.$

39. $\frac{d+5}{2d-3}=\frac{d+2}{2d-1}.$

40. $\frac{18}{x} - \frac{18}{x+3} = 1.$

41. $\frac{120}{x} - 4 = \frac{120}{x+5}.$

42. $\frac{7}{x+3} + \frac{4}{x-2} = \frac{31}{x^2+x-6}.$

43. $\frac{x+2}{x-2} - \frac{x-1}{x^2-4} = -1.$

44. $\frac{4s}{s^2-9} + \frac{1}{3-s} = \frac{5}{s+3}.$

45. $\frac{3y-2}{0.5y-1} = \frac{0.3y+1}{0.05y}.$

46. $\left(\frac{15}{x} - \frac{1}{4}\right)(x+2) = 15.$

47. $\frac{5-x}{2-x} - \frac{x+2}{1-x} - \frac{57}{x^2-3x+2} = 0.$

48. $\frac{y+9}{9-4y^2} + \frac{1}{y} + \frac{1-y}{2y^2+3y} = 0.$

49. $\frac{2x+1}{x^3+8} = \frac{x^2+1}{x^2-2x+4} - \frac{x}{x+2}.$

50. $\frac{17}{12n+24} - \frac{2}{3-n} + \frac{7}{2} = \frac{n}{n-3} + 3 - \frac{2n+9}{4n+8}.$

51. Separate 264 into two numbers whose quotient is 32.

52. The denominator of a fraction is 5 larger than the numerator. If 2 is added to the numerator, the value of the fraction becomes $\frac{3}{4}$. What is the given fraction?

53. The sum of two numbers is 194. If the larger number is divided by the smaller, the quotient is 11 and the remainder is 2. Find the two numbers.

54. If we add 8 to twice a certain number and divide this sum by a number which is 10 less than the original number, the quotient is $2\frac{1}{2}\frac{4}{9}$. Find the number.

55. The number 52 is divided in turn by two different numbers, one of which is 4 larger than the other. The difference between the quotients is $\frac{1}{6}\frac{3}{}$. What are the divisors?

65. Sets of Equations containing Fractions. It is obvious that a set of simultaneous equations involving fractions may be cleared of fractions by the process explained above. The resulting set of equations may then be solved by one of the methods explained in Chapter II.

Exercises. *Sets of Equations containing Fractions*

Solve the following sets of equations, and check each solution:

1. $\frac{3x}{2} + \frac{2y}{3} = 5,$
$\frac{y+1}{4} - \frac{x+1}{6} = \frac{1}{2}.$

2. $5R - \frac{2S}{3} = 2\frac{1}{3},$
$\frac{S-2}{R} - 1 = \frac{2S-7}{R}.$

3. $\frac{n-1}{d} = \frac{1}{4},$
$\frac{n+5}{2d} = \frac{1}{2}.$

4. $\frac{9}{a+b} = -3,$
$\frac{7}{a-b} = -1.$

5. $\frac{c-3}{c+1} = \frac{4}{d+1} + -,$
$\frac{3-2d}{2c+d+2} = 3.$

6. $\frac{2x+y}{x^2-y^2} = \frac{8}{x+y} - \frac{1}{x-y},$
$\frac{x+y+4}{x^2-4xy+3y^2} = \frac{7}{x-3y}.$

7. $0.4a + 0.3b = 1.3,$
$0.6a + 0.2b = 1.2.$

8. $5p - 2q = -0.5,$
$0.8q + 0.4p = 1.$

9. $0.01r + 0.7w = 13.9,$
$0.5r - 0.03w = -5.6.$

10. $0.6c + 10d = 0.28,$
$9c - 0.5d = 2.695.$

11. $\frac{6}{s-4} = \frac{8}{t-3},$
$\frac{1}{s-5} = \frac{3}{t+4}.$

12. $\frac{1}{2}(A+B) = 11,$
$\frac{1}{2}(A-B) = 4.$

2. Problems

66. Problems involving Fractional Equations. The conditions of a verbal problem may be such that they can be represented most conveniently by one or more fractional equations, such as you have been studying.

Illustrative Example. An automobilist drove a distance of 150 mi. Returning, he increased his speed 5 mi. an hour and took 1 hr. less. What was his original rate per hour?

Solution. Let r miles per hour = the original speed. Then $(r + 5)$ miles per hour = the return speed. Hence we obtain the equation

$$\frac{150}{r} - 1 = \frac{150}{r+5}. \quad \text{(Explain.)}$$

(Show by completing the solution that $r = 25$.)

Exercises. *Problems*

Motion Problems

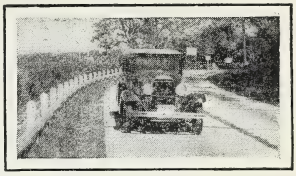
1. By travelling 6 mi. an hour faster, a train would have required $\frac{1}{2}$ hr. less to run 168 mi. What was its rate?

2. It takes a certain freight train 1 hr. 40 min. longer to go 240 mi. than it does a passenger train. The rate of the freight train is three fourths that of the passenger train. Find the rate of each.

3. An aviator made a trip of 242 mi. If he had averaged 20 mi. an hour more, he could have gone 44 mi. farther in the same time. What was his average speed?

4. Two motorists started at the same time from the same place to drive 70 mi. The rate of the first was 3 mi. an hour less than that of the second. When the second motorist had reached the destination, the first lacked 6 mi. of finishing the journey. Find the average rate of each.

5. A motorist, having 50 mi. more to travel, finds that if he continues at his present rate, he will be 45 min. late for an appointment. He also finds that in order to arrive on time he must increase his speed by 15 mi. an hour. What was his rate at first?



Work Problems

6. A certain machine can turn out in 4 hr. a job which a smaller machine would require 6 hr. to complete. How many hours will it take to complete the work if both machines are used together?

Solution. Let h = the number of hours required when both machines are used. The conditions of the problem may be tabulated as follows:

	First Machine	Second Machine
Hours required . . .	4	6
Work done in one hour	$\frac{1}{4}$	$\frac{1}{6}$
Hours worked	h	h
Work done	$\frac{1}{4}h$	$\frac{1}{6}h$

Since the two machines together can do $\frac{1}{h}$ of the work in 1 hr., the total work they will do in h hours is $\frac{1}{h} \cdot h$, or 1. Hence we have the following equation:

$$\tfrac{1}{4}h + \tfrac{1}{6}h = 1.$$

(Complete the solution.)

7. A swimming pool can be filled by two inlet pipes in 4 hr. and $5\frac{1}{4}$ hr. respectively, and emptied by one outlet pipe in 6 hr. The pool was emptied to clean it. Then both

inlet pipes were opened to fill it, but the outlet pipe was left open by mistake. How long, to the nearest minute, did it take to fill the pool?

8. In a newspaper office there are two presses. The daily edition can be printed by the larger press in 72 min. and by the smaller in 90 min. After the larger press had been in use for 10 min. in printing the edition, it broke down and the work had to be transferred to the other press. How long did it take the second press to complete the printing?

Miscellaneous Problems

9. A dealer paid \$360 for a certain number of chairs of the same kind. He sold all but two for \$330 at an advance of \$10 on the cost of each. How many chairs did he buy?

10. Mary was sent to a store to buy as many oranges as possible for \$1. She found that the price per dozen was \$0.12 more than formerly, and that she received 5 fewer oranges for \$1. What was the former price per dozen?

11. A certain fraction becomes equal to $\frac{1}{2}$ if 2 is added to its denominator, and equal to $\frac{3}{4}$ if 1 is added to its numerator. What is the fraction?

Mixture Problems

12. A confectioner wishes to make up pound boxes of candy which he can afford to sell for 75¢ each. If he plans to use 24 lb. of candy which sells for 70¢ a pound, how many pounds of candy selling at 90¢ a pound should he mix with it?

Suggestion. If n = the number of pounds of 90-cent candy used, we have $0.90\,n + 0.70 \times 24 = 0.75(24 + n)$. (Explain.)

13. A grocer has two grades of coffee, a 45-cent grade and a 33-cent grade, which he wishes to blend. How many pounds of each should he use to make a mixture of 100 lb. which he can sell at 42¢ a pound?

14. How much water must be added to 2 qt. of a 15% solution of acid to reduce it to a 10% solution?

Solution. Let n = the number of quarts of water which must be added. The conditions of the problem may be tabulated as follows:

	Original Solution	New Solution
Quantity of solution	2	$2 + n$
Per cent of strength	15%	10%
Amount of acid . .	0.15×2	$0.10(2 + n)$

Since the amount of acid remains the same in both solutions, we have the following equation:

$$0.30 = 0.10(2 + n). \quad \text{(Explain.)}$$

(Complete the solution.)

15. A nurse prepared 5 qt. of a 10% solution of disinfectant. Later she found it necessary to reduce it to a 6% solution. How much water did she have to add?

16. An automobile radiator contains 12 qt. of a 15% mixture of alcohol and water. How much alcohol must be added to make it a 25% solution?

17. How much water must be evaporated from 20 gal. of an 8% solution of salt water to leave a solution which is 12% salt?

18. A dairyman wants to combine milk which contains 4% butter fat and cream which contains 80% butter fat to make 20 gal. of a mixture which will contain 50% butter fat. How many gallons of each must he use?

3. Literal Equations and Formulas

67. Literal Equations with Fractions. We are now ready to study fractional types of literal equations and formulas. The formula may be regarded as a special type of literal equation.

Exercises. *Literal Equations and Formulas containing Fractions*

Solve each of the following equations for x:

1. $\frac{3\,ax}{4} = 15\,a.$
2. $\frac{5\,n}{x} + 1 = 6.$
3. $\frac{k}{6} = \frac{5\,k^2}{4\,x}.$
4. $\frac{l}{2\,x} + 3 = \frac{6 - 4\,l}{x}.$
5. $\frac{d}{2\,x} = \frac{4\,d}{3\,x} - \frac{5}{9}.$
6. $\frac{n}{dx} = \frac{7}{d}.$
7. $\frac{a^2}{x} - b = \frac{b^2}{x} + a.$
8. $\frac{x}{c} + \frac{x}{d} = c + d.$
9. $\frac{1}{m} - \frac{1}{n} = \frac{1}{x}.$
10. $c\left(\frac{d}{ab} - \frac{ab}{x}\right) + d = \frac{c^2}{x}.$
11. $\frac{n+x}{d+x} = 1\tfrac{1}{4}.$
12. $\frac{n+x}{d+x} = \frac{n}{d} + \frac{1}{6}.$
13. $\frac{x-s}{2\,x+t} = \frac{3\,x+t}{6\,x-s}.$
14. $\frac{5-a}{4\,b-x} = \frac{5+a}{4\,b+x}.$
15. $\frac{x+a}{2} - \frac{2}{x+a} = \frac{x-a}{2}.$
16. $\frac{x}{a} = 5 - \frac{6\,a}{x}.$
17. $\frac{2}{5}\left(\frac{x}{r} + \frac{r}{x}\right) = 1.$
18. $\frac{x-n}{n} - \frac{n}{x-n} = \frac{3}{2}.$
19. $\frac{x^2+n^2}{4\,x^2-9\,n^2} = \frac{x}{2\,x+3\,n} - \frac{1}{4}.$
20. $\frac{a}{x} - \frac{b}{cx} = \frac{d}{cx} - \frac{b-a}{c}.$

Solve the following pairs of equations for x and y:

21. $\frac{3x}{2a} - \frac{y}{a} = 3,$
$\frac{x}{a} + \frac{3y}{4a} = 6\frac{1}{4}.$

22. $\frac{x}{c} + \frac{5y}{3c} = \frac{1}{3},$
$\frac{x}{d} + \frac{3y}{2d} = \frac{1}{2}.$

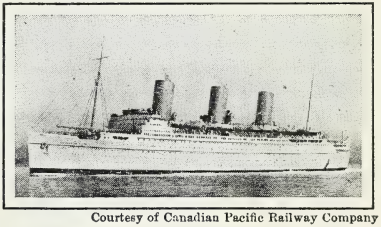
23. $\frac{x-3}{n} = \frac{2-3y}{2m},$
$\frac{3x+2}{3n} = \frac{y+3}{3m}.$

24. $\frac{x+4}{a} = \frac{y-5}{b},$
$\frac{2x-3}{a} = \frac{y+2b}{b}.$

25. $\frac{x}{a} + \frac{y}{b} = a + b,$
$x + y = ab.$

26. $by - ax = 2b,$
$\frac{x}{b} + \frac{y}{a} = \frac{2}{b}.$

*27. On a trip from England to Canada, an ocean liner covered the distance of m miles at an average rate of r miles per hour. On the return trip the ship covered the same distance in h hours less time. Find the average rate (1) on the return trip; (2) for the round trip.



Courtesy of Canadian Pacific Railway Company

28. Find how much water must be evaporated from g gallons of a $b\%$ salt solution so that the remainder will be a $c\%$ solution.

29. A man was hired for n days at m dollars a day. But he was to forfeit r dollars for every day that he was idle. At the end of n days he received d dollars. How many days was he idle?

* Optional.

Solve each formula for the letter, or the letters, indicated:

30. $F = \frac{ms^2}{r}$ for m; for r.

31. $s = \frac{H}{m(t_2 - t_1)}$ for H.

32. $R = \frac{r_1 r_2}{r_1 + r_2}$ for r_1; for r_2.

33. $W = \frac{2000}{3} \times \frac{bd^2}{L}$ for b.

34. $a = \frac{v_t - v_0}{t}$ for t; for v_0.

35. $F = k \cdot \frac{mm'}{r^2}$ for k; for m.

36. $P = \frac{Fs}{t}$ for F; for t.

37. $E = \frac{FD}{(P + x)d}$ for P.

38. $V = \frac{\pi p r^4}{8\,lv}$ for l; for p.

39. $V_0 = \frac{V}{(1 + at)} \cdot \frac{H}{760}$ for H.

40. The following formula is used in finding the focal length of lenses:

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}.$$

(1) Solve for F; for f_1. (2) Find F if $f_1 = 48$ and $f_2 = 80$.

*41. To find the radius (R) of curvature of a portion of a sphere such as a lens or a spherical mirror, one of two methods may be followed:

(a) When the method of reflection is used, the formula for the computation is

$$R = \frac{2\,AS}{L - 2\,S}.$$

(1) Solve this formula for A; for L; for S. (2) Find R when $A = 518$ cm., $L = 21.5$ cm., and $S = 1.77$ cm.

(b) When a spherometer is used, the formula for the computation is

$$R = \frac{L^2}{6\,d} + \frac{d}{2},$$

Find R when $L = 54.4$ mm. and $d = 4.85$ mm.

*Optional.

*42. To determine the cost (C) of making a number (N) of prints by means of a number (n) of electrotypes, where R is the cost of running the press per hour, S is the speed of the press in number of prints per hour, and E is the cost of one electrotype, the following formula is used:

$$C = \frac{NR}{S(1+n)} + En.$$

(1) Solve for R; for S; for E. (2) Find C if $N = 6000$, $R = \$1.50$, $S = 1000$, $E = \$0.40$, and $n = 4$.

*43. The electric current in a simple circuit for two cells is found by the following formulas:

a. Cells in parallel. $i = \dfrac{E}{R + \dfrac{r}{2}}$;

b. Cells in series. $i = \dfrac{2E}{R + 2r}$.

(1) Solve each of these formulas for E; for r. (2) Find i in each case when cells having a voltage (E) of 1.4 volts and an internal resistance (r) of 0.1 ohm are used with an external resistance (R) of 20 ohms.

*44. The length in feet of rope, wire, hose, etc. wound on a reel may be found by the formula $l = 0.13\,nt(a + e)$, where n is the number of coils side by side, t is the number of layers in the coil, a is the diameter in inches over all, and e is the diameter in inches of the centre portion of the reel. (1) Solve the formula for t; for e. (2) Find l if $n = 11$, $t = 5$, $a = 16$ in., $e = 6$ in.

*45. The greatest height (s) attained by an object thrown vertically upward and the time (t) in seconds required to

*Optional.

reach this height are found by the formulas $s = \frac{v^2}{2g}$ and $t = \frac{v}{g}$, respectively, where v is the initial velocity in feet per second, g is the acceleration due to gravity in feet per second per second, and s is the height in feet of the body. Eliminate g from these formulas.

SUGGESTION. Solve one of the formulas for g, and substitute this value of g in the other formula, thus eliminating g. Or, solve each equation for g and equate the resulting values.

*46. The heat (in calories) developed in an electric circuit and the current flowing in the same are given by the formulas $H = \frac{EIt}{4.18}$ and $I = \frac{E}{R}$, respectively, where H is the number of calories, E is the voltage, I is the current in amperes, t is the time in seconds, and R is the resistance in ohms. Eliminate E from these formulas.

*47. The distance through which a body falls from a state of rest and its velocity at the end of this distance are given by the formulas $s = \frac{1}{2}gt^2$ and $v^2 = 2gs$, respectively, where the letters have meanings similar to those given in Exercise 45. Eliminate g from these formulas.

4. Ratio and Proportion

68. Numerical Comparisons; Ratio. Two numbers can be compared by stating *how much greater* or how much smaller one is than the other, or by stating *how many times* one number contains the other. For example, we may say that 12 is *eight more* than 4, or that 12 *contains* 4 *three times*. When we compare two numbers by division, we are using the *ratio* method.

Thus, the ratio of 12 to 4 is $\frac{12}{4}$, or 3, and the ratio of 9 to 4 is $\frac{9}{4}$, or $2\frac{1}{4}$.

*Optional.

In general, the ratio of a to b is the quotient of a divided by b, or $\frac{a}{b}$. The ratio of a to b is also written in the form $a : b$.

Since a ratio represents an indicated division or a fraction, the principles that apply to fractions apply likewise to ratios.

The ratio of two quantities of the *same kind* must be expressed in terms of the *same unit.*

For example, the ratio of 1 hour to 15 minutes is *not* $\frac{1}{15}$ but $\frac{60}{15}$, or 4.

69. Equal Ratios; Proportion. Any fraction, or ratio, has many equivalent forms. For example,

$$\frac{2}{3} = \frac{6}{9} = \frac{8}{12} = \frac{20}{30} = \frac{2n}{3n}, \text{ and so on.}$$

A statement expressing the equality of two ratios is called a proportion.

Thus, $\frac{2}{3} = \frac{6}{9}$ is a proportion. This proportion may also be written in the form $2 : 3 = 6 : 9$, and is then read "2 is to 3 as 6 is to 9."

The first and fourth terms of a proportion are often called the *extremes*, while the second and third terms are called the *means*.

70. The Rule of Proportion. When one of the terms of a proportion is an unknown, it can readily be found by treating the proportion as a *fractional equation* and solving this equation in the usual way.

Thus, if $$\frac{a}{b} = \frac{c}{d},$$

we may multiply both members by the L.C.D., namely, bd.

Then $$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd.$$

Simplifying, we have $ad = bc$, and we can then solve this equation for any desired letter.

Since any proportion may be written in the form $a : b = c : d$, the preceding explanation also suggests the following simple rule, which is called the *Rule of Proportion*:

> *In a proportion, the product of the outer terms (the extremes) equals the product of the inner terms (the means).*

HISTORICAL NOTE. This rule of proportion was known to the Hindus as early as the seventh century as the Rule of Three, but it was stated as an arbitrary rule. Bhāskara, who lived in the twelfth century, stated this rule as follows: "The first and last terms, which are the *argument* and *requisition*, must be of like denomination; the *fruit*, which is of a different species, stands between them; and that, being multiplied by the *demand* (that is, the *requisition*) and divided by the first term, gives the *fruit* of the *demand* (that is, the *Produce*)." Not until about the fifteenth century did mathematicians begin to appreciate the connection of the Rule of Three with proportion.

Exercises. *Using Ratios and Proportions*

1. Write each of the following ratios in two different ways, and simplify the ratios when possible:

(1) 1 to 3; (4) 9 in. to 15 in.; (7) $\frac{1}{2}$ pt. to 1 gal.;
(2) 7 to 4; (5) $1\frac{1}{2}$ ft. to 8 in.; (8) 5 mm. to 2 cm.;
(3) 4 to 6; (6) 15 oz. to 3 lb.; (9) \$2 to 16¢;

(10) $(a^2 - b^2)$ to $(a - b)$; (11) $\left(1 - \frac{3}{x}\right)$ to $\left(1 - \frac{9}{x^2}\right)$.

2. An architect makes the drawings for a house, using the scale of $\frac{1}{4}$ in. to 1 ft. What fractional part of any given dimension of the house is the corresponding dimension of the drawing?

3. A man borrows a sum of money at 6% interest. What is the ratio of the interest paid each year to the principal borrowed?

4. How is the ratio $\frac{9}{6}$ affected by (1) multiplying each term by 3? (2) dividing each term by 3? (3) increasing each term by 3? (4) diminishing each term by 3?

5. Specific gravity is the ratio of the weight of a body to the weight of an equal volume of water. (1 cu. ft. of water weighs approximately 62.5 lb.)

(1) If $\frac{1}{2}$ cu. ft. of common salt weighs 67.8 lb., what is its specific gravity?

(2) The specific gravity of gold is 19.3. Find the weight of 1 cu. in. of gold.

Test the correctness of each of the following proportions:

6. $\frac{4}{3} = \frac{16}{12}$.

7. $\frac{6}{15} = \frac{3}{7\frac{1}{2}}$.

8. $2\frac{1}{2} : 3 = 4\frac{1}{3} : 5$.

9. $\frac{6\,l}{5\,w} = \frac{18\,l}{15\,w}$.

10. $\frac{6(a+b)}{2(a^2-b^2)} = \frac{3}{a+b}$.

11. $\frac{15\frac{2}{3}\,x}{12\,y} = \frac{25\,x}{18\,y}$.

Find the value of x in each of the following proportions:

12. $\frac{7}{9} = \frac{21}{x}$.

13. $\frac{a}{x} = \frac{b}{c}$.

14. $\frac{x-15}{4} = \frac{x-10}{6}$.

15. $\frac{2}{x-1} = \frac{3}{x}$.

16. $\frac{x}{c-d} = \frac{1}{c+d}$.

17. $\frac{1+x}{2+x} = \frac{3+x}{5+x}$.

Write each of the following equations in the form of a proportion:

18. $mn = ax$.

19. $x(1+x) = 3\,a$.

20. $ax - cx = 2\,b$.

21. $x^2 - 4 = a^2 + ac$.

22. In the following table the ratio $h : d$ is constant. Find the missing numbers.

$h =$	1	2	5	
$d =$	40	80		360

71. Solving Problems by Proportion. Many problems arising in business, in the shop, and in science laboratories can be solved quickly by using proportion. In any problem of this type there must be four numbers which can be so arranged that we obtain an equality of ratios in accordance with the conditions of the problem.





Proportion in a photographic enlargement

Whenever we have a situation involving two variables, such as x and y, which increase or decrease together in a manner suggested by the relation $y = mx$, we can apply the Rule of Proportion. This relation may also be written in the form $\frac{y}{x} = m$, where m is constant.

For example, if coffee sells at a price of 40¢ a pound, the cost (c) of any number of pounds (n) is obtained by the formula $c = 40\,n$. That is, the ratio $\frac{c}{n}$ is constant, namely, 40.

Illustrative Example. How many pounds of coffee can be bought for \$9.60, if 5 lb. cost \$2?

Solution. The conditions of the problem may be tabulated as follows, the letters c and n being used in the sense explained above.

Let x = the required number of pounds.

$c =$	2	9.60
$n =$	5	x

Since the ratio $\frac{c}{n}$ is constant, we have

$$\frac{2}{5} = \frac{9.60}{x}.$$

Solving this fractional equation, we find that $x = 24$. (Explain.)

Exercises. *Problems involving Ratio and Proportion*

1. One rectangular room is 12 ft. long and 10 ft. wide. Another rectangular room is 15 ft. 4 in. long and 12 ft. 6 in. wide. What is the ratio of (1) the length of the first to the length of the second? (2) the perimeter of the second to the perimeter of the first? (3) the area of the first to the area of the second?

2. A rectangular rug which is 9 ft. by 12 ft. is placed upon the floor of a room which is 14 ft. square. What is the ratio of the area of the rug to the area of the floor?

3. The sides of a triangle are in the ratio of 2 : 3 : 4, and the perimeter is 45 in. Find the length of each side.

SUGGESTION. Represent the sides by $2n$, $3n$, and $4n$ respectively.

4. Divide \$2700 among four persons in the ratio of 3 : 4 : 5 : 6.

5. A merchant offered to sell marbles at the rate of 100 for 25¢. Three small boys pooled their money and bought 100 marbles. The first contributed 5¢; the second, 8¢; and the third, 12¢. How many of the marbles should each boy receive?

6. The three angles of a triangle are to each other as 2 : 5 : 8. How many degrees are there in each angle of the triangle?

7. Three numbers are in the ratio of 4 : 5 : 7. The sum of the two smaller numbers exceeds the largest number by 18. What are the numbers?

8. Find two numbers whose ratio is 7 : 12, such that $\frac{3}{5}$ of the smaller is 3 greater than $\frac{1}{3}$ of the larger.

9. The perimeter of a triangle is f feet, and the ratio of the sides is $m : n : r$. Find each side.

10. How many gallons of gasoline can be bought for \$0.85, if 12 gal. cost \$2.04?

11. Herbert Jones had saved \$50 for a vacation. He found that the first six days of his vacation cost him \$18.75. At that rate, how long a vacation can he have?

12. The freight rate between two cities 200 mi. apart is 40¢ per 100 lb. How should the 40¢ be divided between two railroads, one of which carries the freight 125 mi. and the other 75 mi.?

13. Two numbers are in the ratio of 3 : 4. If 9 is added to each, the ratio becomes 10 : 13. Find the numbers.

14. If a certain number is added to 2, 6, 3, and 8 in turn, it is found that the ratio of the first sum to the second sum equals the ratio of the third sum to the fourth sum. Find the number.

15. If A gives \$20 to B, the ratio of the money B then has to the money A has left will be 10 : 11. But if B gives \$20 to A, the ratio of the money B has left to the money A then has will be 8 : 13. How much money has each?

16. On a certain map, a line that is $1\frac{1}{4}$ in. long represents a distance of 25 mi. What distance is represented on that map by a line that is $3\frac{1}{2}$ in. in length?

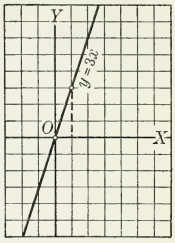
17. If 120 ft. of a certain kind of copper wire weighs 10 lb., what is the weight of 225 ft. of the same kind of wire?

18. If a boat sails m miles in h hours, how many hours will it take to sail d miles at the same rate?

19. If y yards of cloth cost s dollars, how many yards of the same cloth can be bought for d dollars?

5. Direct and Inverse Variation

72. Direct Variation. It has been shown that the graph of any function of the form mx is a straight line passing through the origin. For example, the graph at the right is that of the *function* $3x$ and of the *equation* $y = 3x$.



If a and b are any two values of the variable, $3a$ and $3b$ are the corresponding values of the function. Now $\frac{3a}{3b} = \frac{a}{b}$; that is, the values of the function and variable are so related that their ratio is constant. In such a case we say that the values of one are *directly proportional* to the corresponding values of the other, or that the function *varies directly* as the variable. Such variation of the function is called *direct variation.*

Thus, if $\frac{y}{x} = k$, where k is a constant, then y varies directly as x. This relation may also be indicated by writing $y = kx$.

Note that in direct variation the function is equal to the variable multiplied by a constant. For example, the circumference of a circle is given by the formula $C = 2\pi r$. Since 2π is a constant, it follows that C varies directly as r.

Illustrative Example. Given that y varies directly as x. Suppose that two corresponding values of x and y are 6 and 30 respectively. What is the value of y when $x = 11$?

Solution. 1. Substituting 30 for y and 6 for x in the equation $y = kx$, we have $30 = k \times 6$. That is, $k = 5$.

2. Hence $y = 5x$. When $x = 11$, we have $y = 5 \times 11 = 55$.

Note. If y varies directly as x, the word "directly" is sometimes omitted. In such a case the phrase "varies as" has the same meaning as "varies directly as."

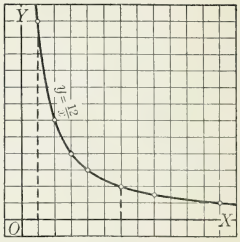
Exercises. *Direct Variation*

1. If y varies as x, and $x = 7$ when $y = 56$, find y when $x = 13$.

2. The velocity (v) of a body falling from rest varies as the time (t) it falls. If the velocity is 192 ft. per second after 6 sec. of falling, find v at the end of 9 sec.

73. Inverse Variation. The following table shows pairs of corresponding values of the variable x and the function $\frac{12}{x}$:

Variable	x	1	2	3	4	6	8	12
Function	$\frac{12}{x}$	12	6	4	3	2	1.5	1



Plotting the function (y) against the variable, we have the graph at the right.

If a and b are any two values of the variable, the corresponding values of the function are $\frac{12}{a}$ and $\frac{12}{b}$. Now $\frac{12}{a} \div \frac{12}{b} = \frac{b}{a}$; that is, any two values of the function are inversely proportional to the corresponding values of the variable. In such a case we say that the function *varies inversely* as the variable, or y varies inversely as x, and such variation of the function is called *inverse variation.*

When one of two variable quantities varies inversely as the other, the quantities are so related that their product is constant. You can easily show that any function of the type $\frac{m}{x}$ varies inversely as x.

In symbols, y varies inversely as x if $xy = k$, or if $y = \frac{k}{x}$.

Exercises. *Inverse Variation*

1. If a triangle has a constant area 100, while its base (b) and height (h) vary, show that the height varies inversely as the base.

2. If a train travels with a speed of s miles per hour and covers 100 miles in t hours, show that the time varies inversely as the speed.

3. At constant temperature the volume (v) of a given mass of gas varies inversely as the pressure (p). Express this law in the "k form."

4. If y varies inversely as x, and $y = 20$ when $x = 4$, find y when $x = 16$.

Solution. 1. Substituting 20 for y and 4 for x in the equation $y = \frac{k}{x}$, we have $20 = \frac{k}{4}$. That is, $k = 80$.

2. Hence $y = \frac{80}{x}$. When $x = 16$, $y = \frac{80}{16} = 5$.

5. In the accompanying table supply the missing values, if y varies inversely as x.

$x =$	2	3	4	—	6	—
$y =$	60	40	—	24	—	15

6. If x varies inversely as y, and $x = 7$ when $y = 12$, find x when $y = 20$.

7. A motorist wishes to drive to a town 160 mi. away from his home. What must be his average speed if he allows for the trip (1) 8 hr.? (2) 4 hr.? Show that the number of hours (h) varies inversely as the average speed (s).

8. Mr. Harris bought 60 sq. yd. of sod for his lawn. If all the sod is to be used in covering a rectangular plot of land, make a table showing possible dimensions of this plot. Show that the length (l) varies inversely as the width (w).

74. Summary. In this chapter you have learned

(1) How to solve fractional equations containing one or more unknowns.

(2) How to solve problems involving fractional equations.

(3) How to solve literal fractional equations.

(4) How to transform formulas containing fractions.

(5) How to use ratios and proportions.

(6) How to solve problems involving ratio and proportion.

(7) How to solve problems involving direct and inverse variation.

Test on Chapter V

Complete correctly each of the following statements:

1. A fractional equation is an equation in which the unknown occurs in the _____ of a fraction.

2. A proportion is a statement showing the _____ of two ratios.

Test the correctness of each of the following proportions:

3. $\frac{3}{4.6} = \frac{8}{11.6}$.

4. $\frac{c^2 - d^2}{4ac - 4ad} = \frac{c+d}{4a}$.

Solve each of the following equations for x:

5. $\frac{2x+1}{3} - \frac{2(1-x)}{6} = \frac{4x-3}{2}$.

6. $0.04(1.2x + 3) - 0.5(0.12x + 1.6) = 0.04$.

7. $\frac{2x+1}{1-2x} + \frac{4x^2+1}{4x^2-1} = \frac{-3}{2x+1}$.

Solve each of the following sets of equations:

8. $\frac{3a-2b}{b+1} = 5$,
$\frac{2a+b}{a-b} = 3$.

9. $x + y = 5$,
$y - z = \frac{3}{2}$,
$x - z = \frac{1}{2}$.

10. Solve the formula $I = \dfrac{E}{\dfrac{R}{n} + r}$ for n.

11. Some members of a club agreed to share equally the expenses of an outing which was to cost \$45. At the last moment five more members joined the party and shared the expenses, so that each person attending paid 45¢ less than the original assessment. How many attended the outing?

12. Find two numbers such that their difference is n and that they are to each other as $a : b$.

13. Show that the number of days (d) required to complete a certain job, if all other conditions remain the same, is inversely proportional to the number of workmen (w). Hence, if 8 men could complete a job in 40 da., how many men would be necessary to complete the job in 20 da.?

Exercises. *Review*

Factor each of the following:

1. $x^{2a} - 14\,x^a + 49$.
2. $x^4 - 4\,x^2 + 3$.
3. $0.0081\,a^4 - 1$.
4. $1 - d^2 + 2\,de - e^2$.
5. $x^3 - 4\,x^2 + x + 6$.
6. $\dfrac{8}{x^3} - 27\,b^6$.
7. $c^5 + 32\,d^5$.
8. $(1 + x^3) + (1 + x)^3$.

9. If $m = \dfrac{4}{5 + \dfrac{1}{r}}$, does m increase or decrease as r increases?

10. Express as a single fraction in its simplest form:

$$\frac{d+c}{d-c} - \frac{2\,d}{c} - \frac{d}{c-d}.$$

11. Indicate in which of the following fractions it is correct to cancel the x's:

(1) $\frac{5x}{x+3}$; (2) $\frac{x+c}{x+2y}$; (3) $\frac{2\,dx}{(c+d)x}$; (4) $\frac{x-(a+b)}{x+5b}$.

12. If the dividend is $4c^2$, the quotient is $2c+3$, and the remainder is 9, find the divisor.

13. A taxicab driver rented a taxicab and agreed to pay the owner \$5 a day plus one fourth of the daily receipts for fares. Write a formula for the total amount (T) that the owner will receive for d days if the driver's receipts average s dollars a day.

14. A house that cost the owner s dollars is rented for " dollars a month. The taxes are d dollars a year, and all other expenses average n dollars a year. What per cent does the investment pay?

15. Given the equation $5x-4y=6$. Find the coördinates of the point at which its graph intersects the y-axis.

A person, to be really "educated," should have been taught the importance of mathematics as an instrument of material conquests and of social organization, and should be able to appreciate the value and significance of an ordered system of mathematical ideas.—Sir T. Percy Nunn

CHAPTER VI

POWERS AND ROOTS

1. Exponents

75. Fundamental Laws of Exponents.* Similar terms involving powers may be combined by *addition* or by *subtraction* without any change in the literal factors. Thus, $10a^3 + 5a^3 - 2a^3 = 13a^3$. The other operations of algebra, however, lead to special laws of exponents, some of which were introduced informally in Chapter I. Each of these laws will now be stated and proved in general form. For the present we shall limit ourselves to *positive integral exponents*. Later, however, we shall have occasion to deal also with exponents which are not integers. It will be seen that the same fundamental laws can be used in the case of all these exponents.

1. *Multiplication Law for Like Bases.* Since $a^3 = a \times a \times a$, and $a^4 = a \times a \times a \times a$, it follows that

$$a^4 \cdot a^3 = a^7.$$

In general, $$a^m \cdot a^n = a^{m+n}.$$

Proof. $a^m \cdot a^n = (a \cdot a \cdot a \ldots \text{to } m \text{ factors})(a \cdot a \cdot a \ldots \text{to } n \text{ factors})$
$= a \cdot a \cdot a \ldots \text{to } (m+n) \text{ factors}$
$= a^{m+n}.$

Hence, the *product of two or more powers having the same base is equal to a power of that base whose exponent is the sum of the exponents of the given powers.*

*Teachers who prefer a closer correlation of exponents and logarithms will find helpful suggestions in the introductory sections of Chapter VIII.

2. *Division Law for Like Bases.* In the division of powers having like bases, three cases may arise. Thus, we may have (1) $a^5 \div a^3$; (2) $a^3 \div a^3$; and (3) $a^3 \div a^5$. Only the first of these cases will now be considered.

We know that

$$\frac{a^5}{a^3} = \frac{\not{a} \cdot \not{a} \cdot \not{a} \cdot a \cdot a}{\not{a} \cdot \not{a} \cdot \not{a}} = a^{5-3} = a^2.$$

In general, $\dfrac{a^m}{a^n} = a^{m-n}$, where $m > n$.*

Proof. $\dfrac{a^m}{a^n} = \dfrac{a \cdot a \cdot a \ldots \text{to } m \text{ factors}}{a \cdot a \cdot a \ldots \text{to } n \text{ factors}}$

$= a \cdot a \cdot a \ldots$ to $(m - n)$ factors (where $m > n$)

$= a^{m-n}$.

State this law in words.

3. *Multiplication Law for Like Exponents.* Observe that

$$a^2b^2 = a \cdot a \cdot b \cdot b = ab \cdot ab = (ab)^2.$$

In general, $a^n \cdot b^n = (ab)^n$.

Proof. $a^n \cdot b^n = (a \cdot a \cdot a \ldots \text{to } n \text{ factors})(b \cdot b \cdot b \ldots \text{to } n \text{ factors})$

$= (ab) \cdot (ab) \cdot (ab) \ldots$ to n factors

$= (ab)^n$.

State this law in words.

4. *Division Law for Like Exponents.* Observe that

$$\frac{a^2}{b^2} = \frac{a \cdot a}{b \cdot b} = \frac{a}{b} \cdot \frac{a}{b} = \left(\frac{a}{b}\right)^2.$$

In general, $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$.

Proof. $\dfrac{a^n}{b^n} = \dfrac{a \cdot a \cdot a \ldots \text{to } n \text{ factors}}{b \cdot b \cdot b \ldots \text{to } n \text{ factors}}$

$= \dfrac{a}{b} \cdot \dfrac{a}{b} \cdot \dfrac{a}{b} \ldots$ to n factors

$= \left(\dfrac{a}{b}\right)^n$.

State this law in words.

* The expression $m > n$ means "m is greater than n," and, similarly, $n < m$ means "n is less than m."

5. *Power of a Power.* Suppose that a^2 is to be raised to the third power. We then have

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^{3\times 2} = a^6.$$

In general, $$(a^m)^n = a^{mn}.$$

Proof.
$$\begin{aligned}(a^m)^n &= a^m \cdot a^m \cdot a^m \cdots \text{ to } n \text{ factors}\\ &= a^{m+m+m+\cdots \text{ to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

State this law in words.

NOTE. Each of these five laws may be written in its *converse* form, as follows:

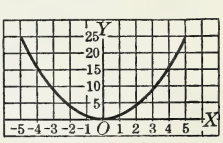
1. $a^{m+n} = a^m \cdot a^n$.
2. $a^{m-n} = \dfrac{a^m}{a^n}$.
3. $(ab)^n = a^n \cdot b^n$.
4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.
5. $a^{mn} = (a^m)^n = (a^n)^m$.

It is suggested that these ten statements be written in the form of a table, which may be used at any time for convenient reference. These laws should be *thoroughly memorized.* Their verbal translation will help considerably in the solution of problems involving exponents.

Exercises. *Using Exponents*

1. Write (1) the *squares* of the first twenty consecutive integers; (2) the *cubes* of the first ten consecutive integers.

2. The diagram at the right shows the graph of $y = x^2$ as x takes on the successive integral values from -5 to $+5$. Make your own table of values for y from the equation, and draw the graph.



3. Draw the graph of $y = x^3$ as x changes successively from -3 to $+3$.

4. Show that $3^3 + 4^3 + 5^3 = 6^3$.

5. Find the value of each of the following expressions:

(1) $3^2 - 2^3$. (3) $6^3 - 6^2$. (5) $6^2 - 2^6$.
(2) $4^3 - 3^4$. (4) $2^5 - 5^2$. (6) $7^2 + 7 \times 2$.

6. Find the value of each of the following expressions:

(1) $(-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 + (-1)^6$.
(2) $(-1)^1 - (-2)^2 + (-3)^3 - (-4)^4 + (-5)^5$.
(3) $4^3 + 3^4 + 2^{10} + (-2)^5 - (-3)^5$.

NOTE. Observe that $(-a)^n$ is a positive or a negative expression according as n is even or odd.

Combine the given expressions as indicated:

7. $(+3)^2 + (-3)^2$.
8. $(-5)^3 + (+5)^3$.
9. $(-8)^2 - (-2)^8$.
10. $(+2)^5 + (-2)^5$.
11. $(-4)^3 - (-3)^4$.
12. $(-3)^5 - (+3)^5$.

13. Express as a power of 10 each of the following: 1000; 10,000; 1,000,000.

14. In the decimal system any number may be expressed as a polynomial involving the successive powers of 10. Thus, $8342 = 8000 + 300 + 40 + 2 = 8 \times 10^3 + 3 \times 10^2 + 4 \times 10^1 + 2$. Write in this form each of the following numbers; 3272; 4035; 620,007; 9340.

15. (*For amusement.*) In an old Egyptian treatise* on mathematics, written by the scribe Ahmes about 1650 B.C., there appeared a problem which may be stated as follows:

In each of seven houses there are seven cats. Each cat kills seven mice. Each mouse would have eaten seven ears of spelt (a kind of wheat). Each ear of spelt would have produced seven *hekat* (measures) of grain. What is the total of all these?

That is, how many measures of grain were saved by the destruction of the mice? Express the result as a power of 7.

*See picture and note about Ahmes Papyrus on page 9.

Exercises. *Applying the Laws of Exponents*

In the following cases perform the indicated operations:

1. $a^4 \cdot a^3$.
2. $x^5 \cdot x^3$.
3. $w^7 \cdot w^6$.
4. $2^5 \cdot 2^4$.
5. $10^4 \cdot 10^6$.
6. $8^3 \cdot 8^5$.
7. $x^{99} \cdot x$.
8. $a^x \cdot ay$.
9. $a^n \cdot a^n$.
10. $a^x \cdot a$.
11. $x^n \cdot x^{2n}$.
12. $x^n \cdot x^2$.
13. $x^4 \cdot x^3 \cdot x^4$.
14. $x^9 \cdot x \cdot x^5$.
15. $n^5 \cdot n^6 \cdot n$.
16. $a^x \cdot a^4 \cdot a^2$.
17. $2^7 \cdot 2^3 \cdot 2^{10}$.
18. $5 \cdot 5^2 \cdot 5^3$.
19. $3\,a^2 \cdot 2\,a^3$.
20. $5\,x^4 \cdot 4\,x^5$.
21. $10\,x^2 \cdot 2\,x^{10}$.
22. $a^4b^2 \cdot a^5n^3$.
23. $a^nb^n \cdot a^2b^3$.
24. $4\,a^x \cdot 5\,a^3$.
25. $(-2\,a^2) \times (3\,a^4)$.
26. $(5\,x^7) \times (-6\,x^3)$.
27. $(4\,a^5) \times (-5\,a^2)$.
28. $(-x)^2 \cdot (-x)^4 \cdot (-x)^5$.
29. $(-a)^5 \cdot (+a)^4 \cdot (-a)^7$.
30. $(a-b)^2 \cdot (a-b)$.
31. $(x-y) \cdot (y-x)^2$.
32. $(a-b)^n \cdot (b-a)^4$.
33. $(a^5+b^2)(a^5-b^2)$.
34. $(a-4\,b^2)^2$.
35. $(3\,a^x-2\,a)^2$.
36. $(x^{2n}-x^ny^n+y^{2n})(x^n+y^n)$.
37. $(x^6+x^4+x^2+1)(x^2-1)$.

38. Multiply: $(5\,a^2-4\,ab+4\,b^2)(3\,a^2+5\,ab-b^2)$.

39. If $2^8 = 256$, and $2^{10} = 1024$, express 256×1024 as a power of 2.

40. Express 452×347 as the product of two polynomials in exponential form.

41. Prove the correctness of the following statement:

$$x^2(x+1)^2 + x^2 + (x+1)^2 = [x^2 + (x+1)]^2.$$

In each case perform the indicated operations:

42. $a^7 \div a^5$.
43. $x^{10} \div x$.
44. $a^{2x} \div a^x$.
45. $2^{10} \div 2^5$.
46. $4^3 \div 4^2$.
47. $a^{5x} \div a^{3y}$.
48. $a^9 \div a$.
49. $a^x \div a$.
50. $a^{x+y} \div a^x$.
51. $36\,a^5 \div 9\,a^3$.
52. $72\,x^5 \div 8\,x$.
53. $(x^{10} + x^8) \div x^3$.
54. $(4\,a^3b^2 - 6\,a^4b^4) \div 2\,ab^2$.
56. $(10\,x^a + 25\,x^{2a}) \div 5\,x^a$.
55. $(6\,x^{4n} - 8\,x^{3n}) \div 2\,x^n$.
57. $(20\,x^{a+b} - 16\,x^{b+c}) \div 4\,x^b$.

Express as a single power each of the following expressions:

58. $5^3 \cdot 2^3$.
59. $25^2 \cdot 4^2$.
60. $a^3 \cdot b^3 \cdot c^3$.
61. $20^3 \cdot 2^3 \cdot 5^3 \cdot 5^3$.
62. $8^2 \cdot 4 \cdot 125^2 \cdot 5^2$.
63. $(-4)^3 \cdot (-2)^3 \cdot (-5)^3$.
64. $(a+b)^2 \cdot (a-b)^2$.
65. $(a-b)^3 \cdot (b-a)^3$.
66. $(3\,x+4\,y)^n \cdot (3\,x-4\,y)^n$.
67. $(a+b)^2 \cdot (a-b)^2 \cdot (a^2-b^2)$.

Perform the indicated operations in each case:

68. $(10^2)^3$.
69. $(3^2)^2$.
70. $[(0.1)^2]^5$.
71. $(a^3)^2$.
72. $(x^4)^5$.
73. $[(-a)^2]^3$.
74. $-(a^2)^3$.
75. $(-2\,a^2)^4$.
76. $(x^a)^b$.
77. $[(-x)^2]^n$.
78. $(x^4y^3)^3$.
79. $(x^ny^n)^2$.
80. $(3\,x^2y)^3$.
81. $(-x^3y^2)^4$.
82. $(-5\,a^2b)^3$.
83. $(a^2b^2c^2)^4$.
84. $(a^nb^nc^n)^x$.
85. $\left(\frac{a}{b}\right)^3$.
86. $\left(\frac{2\,a}{3\,b}\right)^2$.
87. $\left(\frac{4\,x^2}{5\,y}\right)^3$.
88. $(\frac{1}{2}\,ab^2)^3$.
89. $\left(\frac{5\,ab^3}{6\,a^2b}\right)^2$.
90. $\left(\frac{2\,ab^4}{3\,xy^2}\right)^5$.
91. $\frac{(a^3)^2 \cdot (b^2)^2}{(ab)^4}$.
92. $\frac{(x^2y^3)^3}{(x^3y^2)^2}$.

93. If $2^{15} = 32{,}768$, state a convenient method of finding 2^{14}; 2^{16}; 2^{30}.

94. Knowing that $5^4 = 625$, state a convenient method of finding 5^8.

95. Knowing that $5^3 = 125$ and that $3^3 = 27$, how could you find the value of 15^3?

76. Meaning of Zero as an Exponent. In proving the statement $a^m \div a^n = a^{m-n}$, we assumed that $m > n$. But what happens if $m = n$? For example, what is the meaning of $4^3 \div 4^3$?

Obviously, $\frac{4^3}{4^3} = 1$. If we decide, however, to apply the division law stated above in this case also, it is evident that

$$\frac{4^3}{4^3} = 4^{3-3} = 4^0.$$

Hence $$4^0 = 1.$$

In general, we have

$$\frac{a^m}{a^m} = 1, \quad \text{and} \quad \frac{a^m}{a^m} = a^{m-m} = a^0.$$

Hence $$a^0 = 1.$$

That is, *any power having a zero exponent equals 1.**

Thus, $2^0 = 1$; $(a^2)^0 = 1$; $(a - b)^0 = 1$; $(-4\,a^3b^2)^0 = 1$; and so on.

77. Meaning of a Negative Exponent. It is evident that

$$\frac{a^3}{a^5} = \frac{a^3}{a^3 \cdot a^2} = \frac{1}{a^2}.$$

But if we decide to apply the law $a^m \div a^n = a^{m-n}$ in this case also, we have $\frac{a^3}{a^5} = a^{3-5} = a^{-2}$. Hence we are led to the conclusion that $a^{-2} = \frac{1}{a^2}$.

* The expression 0^0, however, has no meaning, since division by zero is excluded.

In general, $a^{-n} = a^{0-n} = \frac{a^0}{a^n} = \frac{1}{a^n}$.

That is, *any quantity with a negative exponent is equal to the reciprocal of that quantity with the corresponding positive exponent.*

In like manner, $\frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = a^n$.

Thus, $x^{-3} = \frac{1}{x^3}$ and $\frac{1}{x^{-3}} = x^3$.

NOTE. When negative exponents occur in a fractional expression, it is necessary to distinguish carefully between *factors* and *terms*.

Thus, $\frac{a^{-2}b^{-3}}{c^{-3}d}$ becomes $\frac{c^3}{a^2b^3d}$.

But $\frac{a^{-2}+b^{-3}}{c^{-3}-d}$ becomes $\frac{\frac{1}{a^2}+\frac{1}{b^3}}{\frac{1}{c^3}-d}$, or $\frac{c^3(a^2+b^3)}{a^2b^3(1-c^3d)}$.

Exercises. *Using Zero and Negative Exponents*

Give the numerical value of each of the following:

1. 4^0, 400^0, 4×10^0, $(4x)^0$, $4^0 \div 5^0$.
2. 7^0, 8×3^0, $5a^0$, $(5x)^0$, $3\left(\frac{2}{3}\right)^0$, $\frac{9}{3^0}$.
3. 10^2, $(-10)^2$, 10^{-2}, $(-10)^{-2}$, 6×10^{-2}.
4. $4^1 - 2^{-1} + 4^{-1} + 9 \times 7^0 + 8 \times 4^{-2}$.
5. $2^{-2} - (\frac{1}{4})^{-2} + (\frac{1}{3})^{-3} - 16^{-1}$.
6. 10×2^{-4}, 8×3^{-3}, $16^2 \times 4^{-2}$, $2^{-4} \times 2^{-3}$, $2^{-2} \times 4^{-2}$, 5^{-2}.
7. $\frac{1}{5^{-2}}$, $\frac{6}{2^{-1}}$, $\frac{9}{3^{-4}}$, $\frac{40^{-3}}{4^{-3}}$, $\frac{5^{-1}}{15^{-1}}$, $\frac{5^{-4}}{5^0}$.
8. $4^{-2} \times 4^{-1}$, $-5^{-3} \times 5^0$, $8^{-5} \div 8^{-4}$, $4^0 \div 6$, 3×5^0, $5^0 \times 2^0$.
9. 6^{-2}, 20^{-1}, $(-3)^3$, $(-3)^{-3}$, $12^0 \times 12^{-2}$, $5^{-4} \times 4^{-4}$.

Express each result with positive exponents:

10. 1^{-a}, a^{-n}, $(w^a)^0$, $a^3 \div a^{-1}$, $x^0 \div y^0$, $a^5 \div a^0$.

11. $(x^{-2})^{-3}$, $(x^{-3})^2$, $(x^{-3})^0$, $(-x^2)^3$, $(-x^2)^{-3}$, $(-x^{-3})^{-2}$.

12. $\left(\frac{a}{b}\right)^{-n}$. [*Answer:* $\left(\frac{b}{a}\right)^{n}$. Note this important result.]

13. $\frac{1}{x^{-1}}$, $\frac{1}{x^{-n}}$, $\frac{x}{y^{-n}}$, $\frac{m^0}{x^{-n}}$, $\frac{1}{a^{-n}}$, $\left(\frac{1}{x}\right)^{-1}$, $\left(\frac{3}{x}\right)^{-2}$.

14. $4\left(\frac{m}{n}\right)^{-1}$, $c\left(\frac{a}{b}\right)^{-1}$, $\frac{1}{a}\left(\frac{b}{a}\right)^{-1}$, $\frac{c^{-4}}{c^{-10}}$, $\frac{x^4}{x^{-3}}$.

15. $\frac{x^{a-3}}{x^{-4}}$, $\frac{x^{-n}}{x^{n-3}}$, $\frac{a^{-3}}{a^{-5}}$, $\frac{b^{-7}}{b^{-4}}$, $\frac{cd^{-3}}{a^{-1}b^0}$.

16. $1^0 + 2^0 + 3^0 + 4^0$.

17. $x^{-3} \cdot a^2 \cdot x^4 \cdot a^{-3}$.

18. $(-a)^2 + (-a)^0 - a^2$.

19. $(-r)^5 \cdot (-r)^{-2}$.

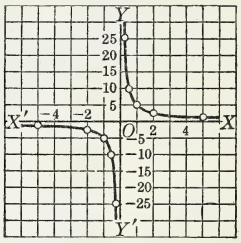
20. $\frac{x^2y^{-4}}{x^{-5}y^{-6}}$.

21. $\frac{3\,x^0y^{-1}z^{-2}}{5\,a^0b^{-1}c^{-2}}$.

NOTE. The graph of $y = 5\,x^{-1} = \frac{5}{x}$ may readily be drawn with the aid of a table such as the following:

$x =$	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$
$y = \frac{5}{x} =$	-1	$-2\frac{1}{2}$	-5	-10	-25

$x =$	$+\frac{1}{5}$	$+\frac{1}{2}$	$+1$	$+2$	$+5$
$y = \frac{5}{x} =$	$+25$	$+10$	$+5$	$+2\frac{1}{2}$	$+1$

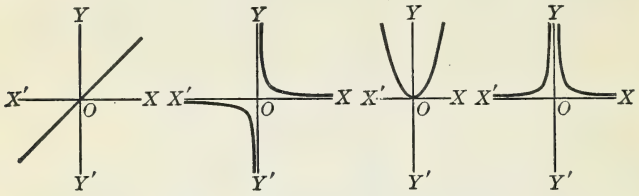


As the value of x becomes smaller, the value of y increases. Similarly, as the value of y becomes smaller, the value of x increases. When $x = +\frac{1}{5}$, $y = +25$; and when $x = -\frac{1}{5}$, $y = -25$. How does y change as x changes from $+\frac{1}{5}$ to $+5$? from $-\frac{1}{5}$ to -5?

Interesting relations exist between the graphs of $y = x$, $y = x^{-1}$, $y = x^2$, $y = x^{-2}$, and so on. These relations are suggested by the graphs

below. Their properties are of importance in many applications of mathematics.

(1) $y = x$. (2) $y = \frac{1}{x}$. (3) $y = x^2$. (4) $y = \frac{1}{x^2}$.



22. Draw each of the graphs pictured above, using any convenient scale.

78. Index Notation.* In certain sciences, especially in physics, chemistry, and astronomy, there is constant occasion to deal with very large and very small numbers. By means of exponents such numbers can be expressed in a very simple and compact form.

For example, one million $= 1{,}000{,}000 = 10^6$;

one billion (a thousand million) $= 1{,}000{,}000{,}000 = 10^9$;

one trillion (a million million) $= 1{,}000{,}000{,}000{,}000 = 10^{12}$.

Likewise, $0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}$.

Also, $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$;

and $0.0000001 = \frac{1}{10{,}000{,}000} = \frac{1}{10^7} = 10^{-7}$.

This method of writing large or small numbers is known as the *index notation* or *exponential notation*. The following exercises will suggest some common applications of this important notation.

* Optional.

Exercises.* *Using the Index Notation*

1. Show that in the index notation 41,000,000 equals 41×10^6.

2. Write in the usual arithmetical form: (1) 2×10^6; (2) 31×10^7; (3) 327×10^9.

3. The distance from the earth to the sun is about 93,000,000 mi. Express this distance in exponential form.

EXPLANATION. Light travels at the rate of about 186,000 mi. a second. The distance which light travels at this rate in the course of a year is called a light-year. A light-year is, very nearly, 5.88×10^{12} miles; that is, nearly 6 trillion miles.

4. The distance from the earth to the nearest fixed star, called "Proxima Centauri," is 4.27 light-years. The distance to the fixed star Sirius is 8.65 light-years, while the Pole Star is at a distance of 44 light-years. With the aid of the explanation given above, express each of these distances in exponential form.

5. Show that 0.00041 equals 41×10^{-5}, or 4.1×10^{-4}.

6. In each of the following cases supply the missing exponent:

(1) $0.000064 = 6.4 \times 10^{(\ )}$.

(2) $0.00000834 = 83.4 \times 10^{(\ )}$.

(3) $\dfrac{5}{100{,}000{,}000} = 5 \times 10^{(\ )}$.

(4) $0.00000003645 = 3.645 \times 10^{(\ )}$.

EXPLANATION. The baffling distances of the stars have as their counterpart the almost infinitely small dimensions prevailing in the realm of light-waves, of atoms and electrons. Definitions of the terms used in the following exercises may be found in any recent science text.

* Owing to the rapid progress of science, some of the numerical data used in the following exercises are likely to be revised at any time. Many interesting uses of the index notation will be found in such treatises as that of Sir James Jeans, entitled "The Universe around Us," published by The Macmillan Company, 1929.

7. The modern telescope has enabled astronomers to estimate that the number of stars in the Milky Way is more than 100,000 million. Express this number in exponential form.

8. The diameter of the galactic system (the Milky Way) has been estimated at 220,000 light-years. The Great Nebula in Andromeda, shown in this picture, is at the distance of 900,000 light-years. The most distant nebula visible in a 100-inch telescope is about 140 million light-years from us. Express each of these distances in exponential form.



9. The sun emits a light of 3.23×10^{27} candle-power. Sirius is 26.3 times as bright as the sun. What is its candle-power?

10. The mass of one hydrogen atom is 1.64×10^{-24} gm. (gram). The mass of an electron is 9.01×10^{-28} gm. Show that the mass of an electron is about $\frac{1}{1820}$ of that of an atom of hydrogen.

11. The average wave-length of red light is 710×10^{-7} cm.; that of certain X-rays is 12×10^{-8} cm. Express each of these numbers as a decimal fraction.

12. The diameter of a molecule of hydrogen is 2.4×10^{-8} cm. Express this number as a decimal fraction.

Exercises. Review

In each of the following exercises perform the indicated operations, and simplify the result:

1. $(x+3)(x-3)(x^2-9)$.
2. $(4a+1)(4a-1)(16a^2+1)$.
3. $(a+b)(a-b)(a^2+b^2)(a^4+b^4)$.
4. $(a^4-a^2b^2+b^4)(a^4+a^2b^2+b^4)$.
5. $(x^2-y^2)^2 \div (x+y)^2$.
6. $(x^3-y^3) \div (x^2+xy+y^2)$.
7. $(x^4+x^2y^2+y^4) \div (x^2-xy+y^2)$.
8. $(a^4-b^4) \div (a-b)$.
9. $\dfrac{3a^2}{2a} \div \dfrac{5a^3b}{4a^2} \cdot 2ab$.
10. $\dfrac{12a^3b^2}{7xy^2} \cdot \dfrac{21x^5y^4}{3ab}$.
11. $\dfrac{l^6(a-b)^4}{b^2(b-a)^3}$.
12. $\left(\dfrac{3d}{2c}\right)^2 \cdot \left(\dfrac{c}{d}\right)^3$.
13. $\left(\dfrac{x-y}{x+y}\right)^4 \cdot \left(\dfrac{x+y}{x-y}\right)^3$.
14. $\left(\dfrac{a^3b}{c^4}\right)^3 \cdot \left(\dfrac{bc^2}{a^3}\right)^2$.
15. $\dfrac{(x-2)(w-3)^3}{(3-w)^3(x-2)^3}$.
16. $\left(\dfrac{x}{y}\right)^n \cdot \left(\dfrac{y}{z}\right)^n \cdot \left(\dfrac{z}{x}\right)^{2n}$.
17. $\dfrac{1}{a^6} - \dfrac{1}{a^4} + \dfrac{1}{a^2}$.
18. $\dfrac{3}{x^6} - \dfrac{4}{x^5} + \dfrac{7}{x^4}$.
19. $\dfrac{a^n+b^n}{a^n-b^n} - \dfrac{a^n-b^n}{a^n+b^n}$.
20. $\dfrac{1-a^2}{a^4} + \dfrac{1-a^2}{a^2} + 1$.
21. $\left(\dfrac{r^2}{9} + \dfrac{rs}{6} + \dfrac{s^2}{4}\right)\left(\dfrac{r}{3} - \dfrac{s}{2}\right)$.
22. $\left(\dfrac{x+y}{z}\right)^3 \cdot \left(\dfrac{x-y}{x+y}\right)^2 \div \left(\dfrac{x-y}{z}\right)^3$.
23. Arrange in the order of magnitude, from smallest to largest: 4^{-1}, 2^{-3}, $(\frac{1}{2})^{-1}$, $(\frac{1}{2})^{-2}$, $(\frac{2}{3})^{-3}$.

24. Arrange according to the descending powers of x: $3x^{-2} + 4 + 7x - 2x^2 + x^{-1}$.

25. Multiply $x^{-2} + 3x^{-1} + 6$ by $x^{-2} - 2x^{-1}$.

26. Multiply $3x^{-2} + 4x^{-3} + 5x^{-4}$ by $x^3 + x^2$.

27. Divide $x^{-2} + 5x^{-1} + 6$ by $x^{-1} + 3$.

28. Multiply $ma + na^{-2}$ by $ma - na^{-2}$.

29. Write with positive exponents:

$$\frac{a}{x^{-3}} + \frac{b}{x^{-2}} + \frac{c}{x^{-1}} + \frac{d}{x^{-1}} + \frac{e}{x^0}.$$

Write with positive exponents, and simplify:

30. $\dfrac{5x^{-1}y^{-2}z^{-3}}{7x^{-3}y^{-4}z^{-5}}$.

31. $\dfrac{3a^{-2}b^3c^{-4}}{15a^3b^{-3}c^4}$.

32. $\dfrac{12x^{-3}y^{-2}z^{-4}}{20x^{-5}y^{-3}z^{-6}}$.

33. $\dfrac{b^{-4}}{c^3} \cdot \dfrac{b^{-3}}{c^{-4}}$.

34. $\dfrac{x^{-2} + y^{-2}}{x^{-2}}$.

35. $\dfrac{a^{-3} - b^{-3}}{a^{-3}b^{-3}}$.

36. $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$.

37. $\dfrac{x^{-3} - y^{-3}}{x^{-1} - y^{-1}}$.

Prove the correctness of the following statements:

38. $\dfrac{3^{-1}}{2^{-1} + 3^{-1}} = \dfrac{2}{5}$.

39. $\dfrac{4^{-2}}{4^{-1} - 2^{-3}} = \dfrac{1}{2}$.

2. Square Root

79. Finding Square Roots by a Graph. Each positive number has *two* square roots which are equal in absolute value but have opposite signs. Thus, the square root of 25 may be either $+5$ or -5. This fact is often indicated by saying that "the square root of 25 is plus or minus 5." The *positive* square root is often called the *principal*

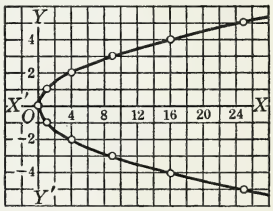
root. In problems involving square root it is usually an easy matter to determine whether both roots are of importance or whether the positive root only is to be used.

NOTE. It is customary to indicate the principal root 5 by $\sqrt{25}$, and the other root -5 by $-\sqrt{25}$. Hence the expression $\pm\sqrt{25}$ means that $+\sqrt{25} = +5$ and $-\sqrt{25} = -5$. When there is no danger of confusion, however, the expression $\sqrt{25} = \pm 5$ is often used in the same sense as $\pm\sqrt{25} = \pm 5$. (See page 175.)

The square roots of positive numbers may be found by a variety of methods. Thus, by drawing the graph of $y = \pm\sqrt{x}$, we may obtain the approximate square root of any number coming within the range of the graph.

To draw the graph of $y = \pm\sqrt{x}$, make a table such as the one shown below. Thus, if $x = 1$, $y = \pm\sqrt{1} = \pm 1$; if $x = 4$, $y = \pm\sqrt{4} = \pm 2$, and so on.

x	$y = \pm\sqrt{x}$
0	0
1	± 1
4	± 2
9	± 3
16	± 4
25	± 5



To find the approximate square root of a number such as 12, locate that number on the horizontal axis and follow the corresponding vertical line to the curve. From the point thus located on the graph, follow the horizontal line to the vertical axis. The value of $\pm\sqrt{12}$ can then be read off approximately on the vertical axis.

80. Finding Square Roots from a Table. The frequent occurrence of powers and roots in mathematical computations has led to the preparation and the constant use of tables from which the desired values may at once be obtained. A portion of a "square root table" is shown on page 311. This table gives the square roots of integers from 1 to 209,

to the nearest thousandth. The way to use this table will be apparent from the following explanation:

$\sqrt{6} = 2.449$ and $\sqrt{60} = 7.746$. To find $\sqrt{65}$, locate 60 in the column headed n, and then follow the same horizontal row to the column headed 5. The number thus located is 8.062, which is the approximate value of $\sqrt{65}$.

Exercises. *Using Square Roots*

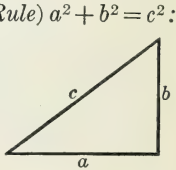
1. Using a larger scale than that in the graph on page 171, draw the graph of $y = \pm\sqrt{x}$, and with the aid of the graph find, to the nearest tenth, the value of $\sqrt{20}$; $\sqrt{27}$; $\sqrt{39}$; $\sqrt{45}$. Verify your results by means of the table given on page 311.

With the aid of the table of square roots, find the square root of each of the following numbers:

2. 7.	4. 85.	6. 103.	8. 199.
3. 80.	5. 15.	7. 176.	9. 205.

If a, b, and c represent the sides of a right triangle, as shown in the figure below, find the missing side in each of the following cases, by applying the formula (Hypotenuse Rule) $a^2 + b^2 = c^2$:

10. $a = 5$, $b = 12$.
11. $a = 15$, $c = 17$.
12. $c = 61$, $a = 60$.
13. $c = 13$, $b = 8$.

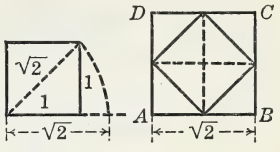


By using the number pairs given below, draw the line AB on squared paper in each of the following cases, and then find its length by the Hypotenuse Rule:

14. A (1, 1); B (4, 5).
15. A (1, 5); B (4, 1).
16. A (2, 10); B (5, 3).
17. A (3, 9); B (7, 0).

3. Radicals

81. Irrational Numbers. Integers and fractions are often called *rational numbers*. For a long time rational numbers seemed to serve every mathematical purpose. But on a certain day, about 2500 years ago, a startling discovery was made. In the school of the great Greek mathematician and philosopher Pythagoras, it was observed that if one draws a square of side 1, and then draws a second square $ABCD$ about it, as shown in the figure above, the area of the square $ABCD$ is *two square units.*



Hence, if the original square represents 1 square inch, the area of $ABCD$ is 2 square inches. Evidently, *the length of AB must be obtained by finding the square root of 2.*

In modern notation, calling this required root n, we have $\sqrt{2} = n$, or $n^2 = 2$. That is, *the length of AB is a number which has 2 for its square.* The Greeks were able to show very clearly that *there is no integer and no fraction whose square is 2.*

Since the value of $\sqrt{2}$ cannot be expressed by means of a rational number, it was called an *irrational* number.

If the ordinary process of finding square roots is applied to $\sqrt{2}$, we obtain an unending decimal; that is, $\sqrt{2} = 1.41421 \cdots$. No matter how far this decimal is extended, *its square never equals 2 exactly.* But we can *approximate* the value 2 as closely as we please, and for practical purposes that is entirely sufficient.

Thus, $(1.4)^2 = 1.96$, $(1.41)^2 = 1.9881$, $(1.414)^2 = 1.999396$.

The number of such irrational numbers is unlimited. Thus $\sqrt[3]{2}$, $\sqrt[4]{6}$, $\sqrt{8}$ are irrational. Many irrational numbers are *not* due to a process of evolution. Thus π, or $3.14159 \cdots$, is an irrational number of a different kind.

82. Definitions. 1. Any expression which contains an indicated root is called a *radical expression* or simply a *radical.*

Since radical expressions occur frequently in important formulas, it is necessary to study the laws which govern their evaluation and their transformation.

Thus, the formula $e = \sqrt[3]{V}$ expresses the edge of a cube in terms of its volume. Similarly, $r = \sqrt{\frac{A}{\pi}}$ gives the radius of a circle in terms of its area, and $c = \sqrt{a^2 + b^2}$ gives the length of the hypotenuse of a right triangle in terms of the other two sides.

The meaning of such terms as *radicand, root, index, radical sign,* was explained in a previous chapter (see page 15).

Thus, in the expression $\sqrt[4]{16} = 2$, 16 is the radicand, 2 is the root, and 4 is the index.

The *order* of a radical is shown by the index of the indicated root.

For example, $\sqrt{5}$ and $\sqrt{xy}$ are of the *second order*, $\sqrt[3]{4}$ is of the *third order*, and so on.

2. Any number which can be expressed as an integer or as the quotient of two integers (positive or negative) is called a *rational number.*

Thus, 8, 3, -2, $\frac{4}{5}$ are rational numbers.

A number that is not rational is said to be *irrational.*

Thus, $\sqrt{5}$, $\sqrt{21}$, π are irrational numbers.

A radical expression such that its indicated root can be found exactly is *rational.*

Thus, $\sqrt{9}$, $\sqrt[3]{27}$, $\sqrt{25\,x^2y^2}$ are rational, while $\sqrt{5}$, $\sqrt{7}$, $\sqrt{20}$ are irrational.

An indicated root of a *positive rational number*, which cannot be found exactly, is often called a *surd*.

For example, $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{17}$ are surds. An expression such as $\sqrt{5}+\sqrt{7}$ is not a surd.

A *quadratic surd* is a surd of the second order. A *binomial quadratic surd* is a binomial which contains at least one quadratic surd.

Thus, $2+\sqrt{3}$ and $\sqrt{5}+\sqrt{11}$ are binomial quadratic surds.

3. It will readily be seen that the square root of an algebraic expression may be positive or negative (see page 170).

Thus, the square roots of $4a^2$ are $+2a$ and $-2a$.

The positive square root of an expression is called its *principal square root*. If no plus or minus sign precedes the radical, the principal root is understood. A radical expression involving a square root is to be combined with other terms by addition or subtraction according as the radical sign is preceded by a plus (+) or a minus (−) sign.

For example, $\sqrt{49}+\sqrt{36}=7+6=13$, and $\sqrt{49}-\sqrt{36}=7-6=1$. That is, $\sqrt{49}+\sqrt{36}$ does *not* mean $\pm 7 \pm 6$, which would lead to the results $+13$ or -13, and $+1$ or -1.

83. The Fundamental Principle of Radicals. It was shown in Chapter I that the operation of finding a root (*evolution*) is the *inverse* of the operation of obtaining a power (*involution*). Moreover, we can check the correctness of a required root by applying the fundamental principle:

When a root is raised to the power expressed by the index, the result must be equal to the radicand.

Thus, $\sqrt[3]{125}=5$, since $5^3=125$. Also, $\sqrt[5]{32}=2$, since $2^5=32$.

In general, if $\sqrt[n]{a}=b$, then $b^n=a$.

From this principle we can readily derive two important relations which are used extensively in the study of radicals.

Observe that $\sqrt{4^2} = 4$, $\sqrt[3]{8^3} = 8$, $\sqrt[4]{a^8} = a^2$, $\sqrt[5]{x^5y^5} = xy$.

Again, $(\sqrt{4})^2 = 4$, $(\sqrt[3]{8})^3 = 8$, $(\sqrt[4]{16})^4 = 16$, $(\sqrt[3]{a^6})^3 = a^6$.

In general, $\sqrt[n]{a^n} = a$, and $(\sqrt[n]{a})^n = a$.

State these relations in words.

Exercises. *Review*

Find the indicated root in each case:

1. $\sqrt{441}$.
2. $\sqrt[3]{343}$.
3. $\sqrt[4]{625}$.
4. $\sqrt{361\,x^2y^4}$.
5. $\sqrt[3]{-1000\,p^3q^6}$.
6. $\sqrt{289\,a^2b^2}$.
7. $\sqrt[3]{-125\,m^3n^3}$.
8. $\sqrt{\frac{2}{3}\frac{5}{6}\,c^2d^6}$.
9. $\sqrt[5]{a^{10}b^{15}}$.

10. Show that $\sqrt[5]{32} - \sqrt[3]{27} + \sqrt[4]{81} - \sqrt[3]{8} = 0$.

11. Show that (1) $\sqrt[n]{0} = 0$; (2) $\sqrt[1]{a} = a$; (3) $\sqrt[n]{1} = 1$.

12. Observe that $2 = \sqrt{2^2} = \sqrt[3]{2^3} = \sqrt[4]{2^4}$, and so on. In like manner, express 3 in radical form, using as indices, in succession, 2, 3, 4, 5.

13. Write as a square each of the following: 7, 15, a, $5\,b$, $(a - b)$, 2^5, $\frac{3}{5}$.

14. Show that $\sqrt{9} + \sqrt{16}$ does *not* equal $\sqrt{9 + 16}$.

15. Show that $\sqrt{a^2 + b^2}$ does *not* equal $a + b$.

Simplify each of the following:

16. $\sqrt[3]{4^3}$; $\sqrt[5]{2^{10}}$; $\sqrt[3]{x^9}$; $(\sqrt{a})^2$; $(\sqrt{x})^4$.
17. $\sqrt{(a + b)^2}$; $(\sqrt{a + b})^2$; $\sqrt[3]{-a^3}$.
18. $(5\sqrt{x})^2 - (3\sqrt{x})^2 + (\sqrt{x})^2 - (4\sqrt{x})^2$.

19. $(5\sqrt{2})^2$; $(4\sqrt{x})^2$; $(x\sqrt{x})^2$; $(x\sqrt[3]{x})^3$; $\sqrt[3]{-x^6}$.

20. $\sqrt{a}\cdot\sqrt{a}$.

21. $\sqrt{3\,n}\cdot\sqrt{3\,n}$.

22. $\sqrt[3]{a}\cdot(\sqrt[3]{a})^2$.

23. $\sqrt[4]{x}\cdot(\sqrt[4]{x})^3$.

24. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$.

25. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.

26. $(5\sqrt{2}+7)(5\sqrt{2}-7)$.

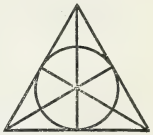
27. $(a+\sqrt{b})(a-\sqrt{b})$.

28. $4(\sqrt{3}+\sqrt{2})-\sqrt{2}(2-\sqrt{2})+\sqrt{3}(\sqrt{3}-4)$.

SUGGESTION. Remove parentheses and collect similar terms.

29. Find the side and area of a square whose diagonal is (1) 2 in.; (2) 8 in.; (3) 10 in.

*30. The side of an equilateral triangle is 10 in. long. Find (1) the altitude; (2) the area of the triangle; (3) the radius of the inscribed circle; (4) the area of the inscribed circle.



84. Fundamental Laws of Radicals. No special laws arise from the *addition* or *subtraction* of radical expressions. Thus, $\sqrt{3}+5\sqrt{3}=6\sqrt{3}$, as in the case of similar terms. The higher operations involving radicals can, however, be performed more readily by the application of definite rules which will now be considered.

1. *The Multiplication Law.* In each of the following cases observe that the radicals have the same index:

(1) $\sqrt{25}\cdot\sqrt{9}=5\times3=15=\sqrt{225}=\sqrt{25\times9}$.

(2) $\sqrt[3]{8}\cdot\sqrt[3]{1000}=2\times10=20=\sqrt[3]{8000}=\sqrt[3]{8\times1000}$.

(3) $\sqrt[3]{27\,a^3}\cdot\sqrt[3]{b^3}=3\,a\cdot b=\sqrt[3]{27\,a^3b^3}=\sqrt[3]{27\,a^3\cdot b^3}$.

These illustrations suggest that, in general,

$$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}, \quad \text{and} \quad \sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$$

*Optional.

That is, *the product of two radicals of the same order is equivalent to the like root of the product of their radicands.* Also, *any indicated root of a product is equivalent to the product of the like roots of its factors.*

This important law enables us to transform, or simplify, a radical whenever its radicand contains a factor whose indicated root can readily be found.

Thus, $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$.

Similarly, $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$.

Likewise, $\sqrt{32\,a^5b^2} = \sqrt{16\,a^4b^2 \cdot 2\,a} = 4\,a^2b\sqrt{2\,a}$.

NOTE. This method of "simplifying" a radical is of particular advantage when a table of roots is available. Thus,

$$\sqrt{300} = 10\sqrt{3} = 10 \times 1.732 = 17.32.$$

(See the table on page 311.)

In like manner, $\sqrt[3]{800} = \sqrt[3]{8 \times 100} = 2\sqrt[3]{100}$. From a table such as the one given on page 311 we find that $\sqrt[3]{100} = 4.642$. Hence $\sqrt[3]{800} = 9.28$ to the nearest hundredth.

2. *The Division Law.* Study the following cases which involve the division of radicals having the same index:

(1) $$\frac{\sqrt{100}}{\sqrt{4}} = \frac{10}{2} = 5 = \sqrt{25} = \sqrt{\frac{100}{4}}.$$

(2) $$\frac{\sqrt[3]{8\,x^3}}{\sqrt[3]{y^3}} = \frac{2\,x}{y} = \sqrt[3]{\frac{8\,x^3}{y^3}}.$$

Hence we infer that, in general,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

State these relations in words in a manner similar to that suggested for the multiplication law.

NOTE. A brief *table of cube roots* is given on page 311. Its use is similar to that of the table of square roots.

Exercises. ***Multiplication and Division with Radicals***

In each of the following cases perform the indicated multiplication or division, and express the result in its simplest form:

1. $\sqrt{50} \cdot \sqrt{2}$.
2. $\sqrt{12} \cdot \sqrt{3}$.
3. $\sqrt[3]{4} \cdot \sqrt[3]{2}$.
4. $\sqrt{18} \cdot \sqrt{2}$.
5. $\sqrt{27} \cdot \sqrt{3}$.
6. $\sqrt[3]{3} \cdot \sqrt[3]{9}$.
7. $\sqrt[3]{4} \cdot \sqrt[3]{16}$.
8. $\sqrt{2} \cdot \sqrt{8}$.
9. $\sqrt{2\,a} \cdot \sqrt{32\,a}$.
10. $\sqrt{8\,b} \cdot \sqrt{8\,b^3}$.
11. $\sqrt{x} \cdot \sqrt{x^5}$.
12. $\sqrt{72} \cdot \sqrt{2\,n^4}$.
13. $\sqrt[3]{5} \cdot \sqrt[3]{25}$.
14. $\sqrt{x^3y} \cdot \sqrt{xy}$.
15. $\sqrt{3\,x} \cdot \sqrt{3\,x}$.
16. $\sqrt[3]{4\,a^2} \cdot \sqrt[3]{2\,a}$.
17. $\sqrt{\frac{1}{2}} \cdot \sqrt{8\,a^2}$.
18. $\sqrt{\frac{3}{5}\,y} \cdot \sqrt{15\,y^3}$.
19. $\sqrt{ab^3} \cdot \sqrt{ab}$.
20. $\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y}$.
21. $\sqrt[n]{a} \cdot \sqrt[n]{a^{2n-1}}$.
22. $\sqrt[x]{r^{2x}} \cdot \sqrt[x]{r^{3x}}$.
23. $\dfrac{\sqrt{20}}{\sqrt{5}}$.
24. $\dfrac{\sqrt{18}}{\sqrt{2}}$.
25. $\dfrac{\sqrt{200}}{\sqrt{8}}$.
26. $\dfrac{\sqrt[3]{16\,a^4}}{\sqrt[3]{2\,a}}$.
27. $\dfrac{\sqrt{108\,x^5}}{\sqrt{3\,x}}$.
28. $\dfrac{\sqrt[3]{625\,a^6}}{\sqrt[3]{5\,b^3}}$.
29. $\dfrac{\sqrt{28\,n}}{\sqrt{7\,n^3}}$.
30. $6\sqrt[3]{\frac{8}{27}}$.
31. $6\,y\sqrt{\dfrac{x^2}{9\,y^2}}$.
32. $12\sqrt{\dfrac{25\,n^4}{36}}$.

Transform, or simplify, each of the following radicals:

33. $\sqrt{24}$.
34. $\sqrt{60}$.
35. $\sqrt{18}$.
36. $\sqrt{80}$.
37. $\sqrt{150}$.
38. $\sqrt{180}$.
39. $\sqrt[3]{270}$.
40. $\sqrt[3]{5000}$.
41. $\sqrt[3]{200\,a^2}$.
42. $\sqrt[3]{135\,n}$.
43. $\sqrt{192\,x^3y}$.
44. $\sqrt[3]{375\,ab^3}$.
45. $\sqrt{32\,a^4b}$.
46. $\sqrt{45\,c^2d}$.
47. $\sqrt{75\,rs^2}$.
48. $\sqrt[3]{16\,ab^3}$.
49. $5\sqrt{72\,x^5}$.
50. $2\,a\sqrt{125\,a^2}$.

85. Simplifying Radical Expressions involving Fractions. It is sometimes necessary to find the decimal value of a numerical radical such as $\sqrt{\frac{3}{7}}$. In such a case the result may be obtained in several ways, as shown below.

1. $\sqrt{\frac{3}{7}} = \sqrt{0.428571\cdots} = 0.6546\cdots$.

2. $\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{1.7320\cdots}{2.6457\cdots} = 0.6546\cdots$.

3. $\sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7}\cdot\frac{7}{7}} = \sqrt{\frac{21}{49}} = \frac{\sqrt{21}}{7} = \frac{4.5825\cdots}{7} = 0.6546\cdots$.

Observe that in the third method the given radicand, $\frac{3}{7}$, was transformed into the equivalent fraction $\frac{21}{49}$, thus making it possible to find the indicated root of the denominator. This process, by which the denominator is made a rational number, is called *rationalizing the denominator*. It is the most convenient method of obtaining the desired decimal value of $\sqrt{\frac{3}{7}}$, since it requires merely the division of a decimal by an integer.

In dealing with radicals it is usually found convenient to "simplify" them in such a way that (1) *the radicand is an integer*, (2) *the radical appears only in the numerator and contains no factor whose indicated root can be found exactly.*

Exercises. *Simplifying Radicals containing Fractions*

In each of the following cases explain the method used in simplifying the given radical:

1. $\sqrt{\frac{x}{y}} = \sqrt{\frac{xy}{y^2}} = \frac{\sqrt{xy}}{\sqrt{y^2}} = \frac{\sqrt{xy}}{y} = \frac{1}{y}\sqrt{xy}$.

2. $\frac{10}{\sqrt{5}} = \frac{10\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{10\sqrt{5}}{\sqrt{25}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$.

3. $\sqrt{\frac{3\,x}{5\,y}} = \sqrt{\frac{3\,x \cdot 5\,y}{25\,y^2}} = \frac{1}{5\,y}\sqrt{15\,xy}.$

4. $\sqrt{\frac{5\,c}{d^5}} = \sqrt{\frac{5\,cd}{d^6}} = \frac{1}{d^3}\sqrt{5\,cd}.$

5. $\sqrt{\frac{3\,a}{b^3}} = \sqrt{\frac{3\,ab}{b^4}} = \frac{1}{b^2}\sqrt{3\,ab}.$

Simplify each of the following radicals:

6. $\sqrt{\frac{1}{2}}$.
7. $\frac{12}{\sqrt{3}}$.
8. $\frac{20}{\sqrt{5}}$.
9. $\frac{4\sqrt{5}}{\sqrt{7}}$.
10. $\sqrt{\frac{5}{6}}$.
11. $\frac{2\,a}{\sqrt{3}}$.
12. $2\sqrt{\frac{5}{32\,n}}$.
13. $a\sqrt{\frac{b}{a}}$.
14. $\sqrt{\frac{5}{8}}$.
15. $\frac{x}{\sqrt{x}}$.
16. $\sqrt{\frac{8\,b}{12}}$.
17. $\sqrt{2\frac{1}{7}}$.
18. $6\sqrt{\frac{1}{3}}$.
19. $\sqrt{\frac{3\,x}{11}}$
20. $\sqrt{\frac{a}{b}}$.
21. $\sqrt{\frac{a-b}{a+b}}$.

Find the decimal value of each of the following radicals correct to the nearest hundredth:

22. $\sqrt{\frac{1}{2}}$.
23. $\sqrt{\frac{2}{3}}$.
24. $\sqrt{\frac{5}{8}}$.
25. $\sqrt{\frac{9}{32}}$.
26. $\sqrt{\frac{3}{2}}$.
27. $\sqrt{\frac{5}{18}}$.
28. $\frac{\sqrt{3}}{\sqrt{15}}$.
29. $\frac{2}{\sqrt{2}}$.
30. $\sqrt{\frac{7}{50}}$.
31. $\sqrt{5\frac{1}{2}}$.
32. $\frac{\sqrt{11}}{\sqrt{3}}$.
33. $\frac{\sqrt{5}}{\sqrt{12}}$.
34. $\frac{1}{\sqrt{3}}$.
35. $9\sqrt{\frac{2}{3}}$.
36. $1 \div \sqrt{\frac{1}{2}}$.
37. $1 \div \sqrt{\frac{2}{5}}$.
38. $\frac{6\sqrt{2}}{\sqrt{3}}$.
39. $\frac{15}{2\sqrt{2}}$.

86. Addition and Subtraction of Radicals. When two radicals are of the same order and have the same radicand, they are said to be *similar*.

Thus, $5\sqrt{2}$ and $7\sqrt{2}$ are similar.

Only similar radicals can be combined by addition or subtraction. But dissimilar radicals of the same order can often be transformed into similar radicals by the methods explained previously.

Illustrative Example 1. Combine $7\sqrt{6}+2\sqrt{6}$.

Solution. $7\sqrt{6}+2\sqrt{6}=9\sqrt{6}$.

Illustrative Example 2. Combine $\sqrt{12}-\sqrt{\frac{1}{3}}$.

Solution. $\sqrt{12}-\sqrt{\frac{1}{3}}=2\sqrt{3}-\frac{1}{3}\sqrt{3}=1\frac{2}{3}\sqrt{3}$.

Exercises. *Adding and Subtracting Radicals*

In the following exercises combine the radicals as indicated:

1. $7\sqrt{2}-5\sqrt{2}$.
2. $8\sqrt{3}-6\sqrt{3}$.
3. $4\sqrt{6}+\sqrt{24}$.
4. $3\sqrt{5}-7\sqrt{45}$.
5. $\sqrt{18}+\sqrt{32}$.
6. $3\sqrt{27}-4\sqrt{3}$.
7. $5\sqrt{2}-\sqrt{50}$.
8. $\sqrt{\frac{1}{2}}+3\sqrt{2}$.
9. $\sqrt{\frac{3}{5}}+4\sqrt{15}$.
10. $6\sqrt{40}-5\sqrt{90}$.
11. $4\sqrt[3]{-72}-\sqrt[3]{9}$.
12. $4\sqrt{5}-3\sqrt{\frac{1}{5}}$.
13. $4\sqrt{\frac{1}{2}}+6\sqrt{\frac{1}{8}}$.
14. $\sqrt{x}+5\sqrt{x}$.
15. $2\sqrt[3]{4}-\sqrt[3]{32}$.
16. $\sqrt[3]{54}-4\sqrt[3]{128}$.
17. $\sqrt{8x}+\sqrt{32x}$.
18. $\sqrt[3]{n}+5\sqrt[3]{n}$.
19. $5\sqrt{ab}-\sqrt{4ab}$.
20. $8\sqrt{a^3}-4a\sqrt{a}$.
21. $\sqrt[3]{5x^2}+\sqrt[3]{40x^2}$.
22. $\sqrt{3a^2}-\sqrt{27a^2}+5a\sqrt{3}$.
23. $5\sqrt[3]{3}+4\sqrt[3]{-24}-6\sqrt[3]{81}$.
24. $3\sqrt{150}-\sqrt{\frac{1}{6}}+9\sqrt{\frac{2}{3}}$.

87. Multiplication of Radicals (Polynomials). By applying the usual rules of multiplication and the principles of radicals explained above, it is possible to find the product of polynomials involving radicals.

Illustrative Example. Multiply: $(7\sqrt{2}-4\sqrt{3})(3\sqrt{2}+2\sqrt{3})$.

Solution.

$$\begin{array}{r} 7\sqrt{2}-4\sqrt{3} \\ 3\sqrt{2}+2\sqrt{3} \\ \hline 42-12\sqrt{6} \\ -24+14\sqrt{6} \\ \hline 18+2\sqrt{6} \end{array}$$

The product of $2\sqrt{3}$ and $-4\sqrt{3}$, which is -24, is placed in the integer column under 42.

Exercises. *Multiplication of Radicals*

Write each of the following products in its simplest form:

1. $(\sqrt{18}+2\sqrt{8}+\sqrt{50})\sqrt{2}$.
2. $(5\sqrt{27}-3\sqrt{12}+\sqrt{75})\sqrt{3}$.
3. $(6\sqrt{2}-3\sqrt{6}+5\sqrt{20})3\sqrt{2}$.
4. $(3\sqrt{5}-6\sqrt{10}+8\sqrt{40})2\sqrt{5}$.
5. $(\sqrt{b}+6\sqrt{3\,b}-5\sqrt{8\,b})\sqrt{2\,b}$.
6. $(\sqrt[3]{4}-5\sqrt[3]{12})\sqrt[3]{2}$.
7. $(\sqrt{x}-\sqrt{3\,x})\sqrt{x}$.
8. $(\sqrt{a}+\sqrt{b})^2$.
9. $(\sqrt{a}-\sqrt{2\,b})^2$.
10. $(\sqrt{3}+\sqrt{2})^2$.
11. $(5+2\sqrt{5})(3-\sqrt{5})$.
12. $(\sqrt{11}+\sqrt{6})(\sqrt{11}-\sqrt{6})$.
13. $(\sqrt{5+2\,x}+\sqrt{5-2\,x})^2$.
14. $\left(\sqrt{a}+\dfrac{1}{\sqrt{a}}\right)^2$.
15. $\left(\sqrt{\dfrac{3\,x}{5\,y}}-\sqrt{\dfrac{y}{5\,x}}\right)^2$.
16. $(\sqrt{a-b}-\sqrt{b-c})^2$.

88. Division of Radicals (Polynomials). The division of a polynomial by a *monomial* in radical form can be performed in accordance with the rules previously explained.

Illustrative Example 1. Divide $\sqrt{30} - 6\sqrt{15} + \sqrt{40}$ by $\sqrt{5}$.

Solution. Dividing each term in succession by $\sqrt{5}$, we have

$$\sqrt{6} - 6\sqrt{3} + \sqrt{8} = \sqrt{6} - 6\sqrt{3} + 2\sqrt{2}.$$

2. When the divisor is a *binomial quadratic surd,* it is possible to simplify the work by applying the *Fundamental Principle of Fractions,* as shown below.

NOTE. Two binomials of the form $a + b$ and $a - b$ are said to be *conjugate* binomials. In like manner, $\sqrt{3} + \sqrt{2}$ and $\sqrt{3} - \sqrt{2}$, or $7 + 3\sqrt{5}$ and $7 - 3\sqrt{5}$ are conjugates. Now, when the two conjugate expressions are binomial quadratic surds, it can readily be shown that their product is rational. Thus, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$. This fact enables us to rationalize the denominator of a fraction when it has the form of a binomial quadratic surd.

Illustrative Example 2. Rationalize the denominator of the fraction

$$\frac{\sqrt{5} + 2\sqrt{3}}{2\sqrt{5} - \sqrt{3}}.$$

Solution. We may multiply the numerator and the denominator by the factor $2\sqrt{5} + \sqrt{3}$, which is the conjugate of the denominator. Hence we have

$$\frac{\sqrt{5} + 2\sqrt{3}}{2\sqrt{5} - \sqrt{3}} = \frac{\sqrt{5} + 2\sqrt{3}}{2\sqrt{5} - \sqrt{3}} \times \frac{2\sqrt{5} + \sqrt{3}}{2\sqrt{5} + \sqrt{3}}$$

$$= \frac{16 + 5\sqrt{15}}{20 - 3} = \frac{16 + 5\sqrt{15}}{17}.$$

Exercises. *Division of Radicals*

Divide:

1. $\sqrt{12} + 4\sqrt{6} - 5\sqrt{18}$ by $\sqrt{2}$.
2. $\sqrt{24} + \sqrt{18} - 6\sqrt{21}$ by $\sqrt{3}$.
3. $\sqrt[3]{32} - 8\sqrt[3]{24} + 7\sqrt[3]{60}$ by $\sqrt[3]{4}$.
4. $15\sqrt{60} + 20\sqrt{42} - 45\sqrt{72}$ by $5\sqrt{6}$.
5. $\sqrt{a^3b^5} - 7\sqrt{ab^3} + 9\sqrt{a^5b}$ by $\sqrt{ab}$.

Rationalize the denominator of each of the following fractions

6. $\dfrac{4}{\sqrt{3}+\sqrt{2}}$.

7. $\dfrac{10}{\sqrt{5}-\sqrt{3}}$.

8. $\dfrac{20}{2\sqrt{3}-\sqrt{2}}$.

9. $\dfrac{32}{6-2\sqrt{5}}$.

10. $\dfrac{12\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$.

11. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

12. $\dfrac{4-\sqrt{3}}{4+\sqrt{3}}$.

13. $\dfrac{2\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$.

14. $\dfrac{10-\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

15. $\dfrac{6+\sqrt{2}}{4-3\sqrt{2}}$.

16. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

17. $\dfrac{3\sqrt{a}-4\sqrt{b}}{\sqrt{a}+2\sqrt{b}}$.

Find the value of each of the following expressions, correct to the nearest hundredth:

18. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$.

19. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

20. $\dfrac{3-\sqrt{5}}{2+\sqrt{5}}$.

21. $\dfrac{24}{\sqrt{6}-2\sqrt{3}}$.

22. $\dfrac{2-\sqrt{10}}{\sqrt{5}-\sqrt{3}}$.

23. $\dfrac{8\sqrt{5}-\sqrt{2}}{4\sqrt{3}-3\sqrt{5}}$.

4. Fractional Exponents

89. Meaning of Fractional Exponents. We have seen that a power such as a^m has a definite meaning when m is a positive or a negative whole number, or is equal to 0.

Thus, $a^3 = a \cdot a \cdot a$, $a^0 = 1$, and $a^{-2} = \dfrac{1}{a^2}$.

What meaning will $a^{\frac{1}{2}}$, $a^{\frac{3}{4}}$, etc. have if we apply to them the preceding laws? It will appear that such an extension of the laws of exponents leads to no contradictions.

For example, applying the law $a^m \cdot a^n = a^{m+n}$, we have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

But $\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a.$

That is, $a^{\frac{1}{2}}$ *has the same meaning as* $\sqrt{a}$, since we obtain the same result, a, by multiplying either $a^{\frac{1}{2}}$ or $\sqrt{a}$ by itself.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$

But $\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = \sqrt[3]{a^3} = a.$

That is, $a^{\frac{1}{3}}$ *has the same meaning as* $\sqrt[3]{a}$.

We are thus led to the conclusion that, in general,

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

Again, $a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} = a^{\frac{12}{4}} = a^3.$

But $\sqrt[4]{a^3} \cdot \sqrt[4]{a^3} \cdot \sqrt[4]{a^3} \cdot \sqrt[4]{a^3} = \sqrt[4]{a^{12}} = a^3.$

That is, $a^{\frac{3}{4}}$ has the same meaning as $\sqrt[4]{a^3}$.

Also, assuming the law $a^{mn} = (a^m)^n$ to hold for fractional exponents, we have

$$a^{\frac{3}{4}} = (a^{\frac{1}{4}})^3 = (\sqrt[4]{a})^3.$$

From illustrations of this type we infer that, in general,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \quad \text{and that} \quad a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m.$$

That is, the *numerator* of a fractional exponent indicates the *power* to which the base is to be raised, while the *denominator* indicates the *root* which is to be found.

Observe that *these two processes can be performed in either order without affecting the result.*

For example, $27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9$, or $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9.$

We now see that there is no distinction in use between exponents that are positive integers and any other exponents. Thus a *fractional negative exponent* indicates the *reciprocal* of a number which has a fractional positive exponent.

For example, $a^{-\frac{5}{2}} = \frac{1}{a^{\frac{5}{2}}} = \frac{1}{\sqrt{a^5}}$ or $\frac{1}{(\sqrt{a})^5}$, and $\frac{1}{a^{-\frac{1}{2}}} = a^{\frac{1}{2}}.$

Illustrative Example. Using the table of square roots on page 311 find $2^{\frac{1}{2}}$, $2^{\frac{3}{2}}$, $2^{-\frac{1}{2}}$. (See graph on page 188.)

Solution.

$$2^{\frac{1}{2}} = \sqrt{2} = 1.414.$$

$$2^{\frac{3}{2}} = 2 \times 2^{\frac{1}{2}} = 2(1.414) = 2.828.$$

$$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \tfrac{1}{2}\sqrt{2} = 0.707.$$

90. Using Fractional Exponents. It can be shown that the interpretation of fractional exponents given above is in harmony with all the laws of exponents and radicals that have been used previously.

For example, $a^{\frac{1}{2}} \cdot b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$, just as $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

Again, $(a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}$, just as $(\sqrt{a})^3 = \sqrt{a} \cdot \sqrt{a} \cdot \sqrt{a} = \sqrt{a^3}$.

Hence, in any given problem involving roots we may use either the radical form or the form of a fractional exponent, whichever is found more convenient.

Thus,
$$\sqrt[m]{\sqrt[n]{a}} = \left(a^{\frac{1}{n}}\right)^{\frac{1}{m}} = a^{\frac{1}{mn}} = \sqrt[mn]{a}.$$

Again,
$$\sqrt[3]{x^{-1}\sqrt{y^3}} \div \sqrt{y\sqrt[3]{x}} = \left(x^{-1}y^{\frac{3}{2}}\right)^{\frac{1}{3}} \div \left(yx^{\frac{1}{3}}\right)^{\frac{1}{2}}$$

$$= \frac{x^{-\frac{1}{3}}y^{\frac{1}{2}}}{y^{\frac{1}{2}}x^{\frac{1}{6}}} = \frac{1}{x^{\frac{1}{3}}x^{\frac{1}{6}}} = \frac{1}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{x} = \frac{\sqrt{x}}{x}.$$

HISTORICAL NOTE. As early as 225 B.C., the Greek mathematician Archimedes understood the law of exponents which was stated above in the form $a^m \cdot a^n = a^{m+n}$. The famous *Sand Reckoner*, an essay addressed by Archimedes to Gelon, the King of Syracuse, attempted to prove that the number of grains in a heap of sand as large as the entire universe can be expressed by our number system as 10^{63}. (This exponential notation was, of course, not known to Archimedes.)

The French clergyman Nicole Oresme used *fractional* exponents in the fourteenth century in his treatise *Algorismus proportionum*. Stifel, a German writer of the sixteenth century, developed the idea of *negative* exponents. In the seventeenth century the English mathematician John Wallis explained the general theory of exponents, including those that were both negative and fractional.

Exercises. *Using Fractional Exponents*

Numbers 1 to 11, oral

Express with radical signs:

1. $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, $b^{\frac{1}{4}}$, $b^{\frac{3}{4}}$, $c^{\frac{1}{5}}$, $c^{\frac{2}{5}}$, $x^{\frac{1}{n}}$, $x^{\frac{n}{2}}$.

2. $x^{\frac{3}{2}}$, $x^{\frac{5}{4}}$, $a^{\frac{5}{3}}$, $a^{\frac{5}{6}}$, $b^{1\frac{1}{2}}$, $c^{0.3}$, $(a+b)^{\frac{1}{2}}$.

3. $2\,b^{\frac{1}{2}}$, $4\,b^{\frac{1}{3}}$, $6\,a^{\frac{2}{3}}$, $8\,c^{\frac{4}{5}}$, $2\,x^{\frac{3}{2}}$, $(a^2-b^2)^{\frac{1}{2}}$.

4. $a^{-\frac{1}{2}}$, $a^{-0.5}$, $2\,b^{-\frac{2}{3}}$, $c^{-\frac{1}{4}}$, $5\,c^{-\frac{3}{4}}$, $x^{-\frac{1}{n}}$, $8\,x^{-\frac{a}{b}}$.

Express with fractional exponents:

5. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt{x^3}$, $\sqrt{x^5}$, $\sqrt[4]{a^{-1}}$, $\sqrt[5]{a^2}$, $\sqrt[n]{a}$.

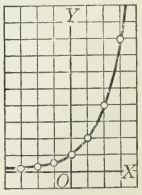
6. $\sqrt[3]{a^4}$, $9\sqrt[4]{a^3}$, $\sqrt[3]{a^5}$, $3\sqrt[5]{a^{-3}}$, $\sqrt[4]{a^2}$, $\sqrt{a^5}$, $x\sqrt[n]{a^b}$.

7. $\sqrt[3]{a^2b^2}$, $\sqrt{a^3b^3}$, $\sqrt{a^{-n}}$, $6\sqrt[4]{x^{-3}}$, $a\sqrt[3]{x}$, $\sqrt[3]{a^{-2}}$, $\sqrt{b^3}$.

Find the numerical value of each of the following:

8. $144^{\frac{1}{2}}$.
9. $25^{\frac{3}{2}}$.
10. $64^{\frac{2}{3}}$.
11. $16^{\frac{3}{4}}$.
12. $\sqrt[3]{27^4}$.
13. $\sqrt[5]{32^4}$.
14. $\sqrt[3]{(-27)^2}$.
15. $(-125)^{\frac{2}{3}}$.
16. $(-32)^{\frac{4}{5}}$.
17. $343^{\frac{2}{3}}$.
18. $\left(\frac{4}{9}\right)^{\frac{1}{2}}$.
19. $\left(-\frac{2}{6}\frac{7}{4}\right)^{\frac{2}{3}}$.
20. $(0.16)^{\frac{1}{2}}$.
21. $(1.69)^{\frac{1}{2}}$.
22. $(0.125)^{\frac{2}{3}}$.
23. $(0.027)^{\frac{2}{3}}$.

24. Draw the graph of $y=2^x$, using in succession the following values of x: $-3, -2, -1, 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 3$.



SUGGESTION. Complete the following table of corresponding values of x and y, and then plot the required graph.

$x=$	-3	-2	-1	0	$\cdots$
$y=2^x=$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\cdots$

25. From the graph suggested in the preceding exercise find the approximate value of y when $x=2\frac{1}{4}$.

26. Draw the graph of $y = 3^x$.

27. Draw the graph of $y = 10^x$, using any convenient scale on the y-axis.

Perform the indicated multiplications:

28. $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$.
29. $x^{\frac{1}{3}} \cdot x^{\frac{2}{3}}$.
30. $y^{\frac{1}{2}} \cdot y^{-\frac{1}{4}}$.
31. $a^{\frac{1}{2}} \cdot a^{\frac{3}{2}}$.
32. $n^{\frac{1}{2}} \cdot n^{\frac{1}{3}}$.
33. $n^{\frac{1}{4}} \cdot n^{\frac{1}{3}}$.
34. $x \cdot x^{\frac{1}{2}}$.
35. $2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$.
36. $3^{\frac{1}{2}} \cdot 3^{-\frac{1}{4}}$.
37. $8^{-\frac{1}{3}} \cdot 8^{\frac{4}{3}}$.
38. $x^{\frac{4}{3}} \cdot x^{-\frac{2}{3}}$.
39. $x^{\frac{1}{5}} \cdot y^{\frac{1}{5}}$.

Perform the indicated divisions:

40. $a^{\frac{3}{4}} \div a^{\frac{1}{4}}$.
41. $x^{\frac{2}{5}} \div x^{\frac{1}{5}}$.
42. $b \div b^{-\frac{1}{2}}$.
43. $x^0 \div x^{-\frac{1}{3}}$.
44. $n^{\frac{3}{5}} \div n^{-\frac{1}{5}}$.
45. $a^{\frac{3}{4}} \div a^{\frac{2}{3}}$.
46. $a^{\frac{3}{4}} \div b^{-\frac{3}{4}}$.
47. $a^{\frac{3}{5}} \div d^{\frac{3}{5}}$.
48. $x^{\frac{1}{2}} \div x^{\frac{1}{3}}$.

In each case perform the indicated operations, writing the result with positive exponents:

49. $a^{\frac{1}{2}} \cdot \sqrt{a}$.
50. $64^{\frac{1}{2}} \cdot 64^{\frac{2}{3}}$.
51. $x^0 \cdot x^{-\frac{1}{2}}$.
52. $4^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$.
53. $3^{\frac{1}{3}} \cdot 9^{\frac{1}{3}}$.
54. $2^{\frac{1}{3}} \div 2^{\frac{1}{4}}$.
55. $a^{\frac{1}{2}} \div \sqrt{a^3}$.
56. $72^{\frac{1}{2}} \div 18^{\frac{1}{2}}$.
57. $\sqrt{a} \cdot \sqrt[3]{a}$.
58. $\sqrt{a} \div \sqrt[3]{a}$.
59. $2^{\frac{1}{4}} \div 32^{\frac{1}{4}}$.
60. $128^{\frac{1}{5}} \div 4^{\frac{1}{5}}$.
61. $(25^{\frac{5}{6}})^3$.
62. $27^{\frac{3}{4}} \cdot 3^{\frac{3}{4}}$.
63. $(a^{-\frac{3}{4}})^{-\frac{4}{5}}$.
64. $(a^4)^{\frac{1}{2}}$.
65. $(x^9)^{\frac{2}{3}}$.
66. $(n^{-8})^{\frac{1}{2}}$.
67. $(a^{-\frac{1}{2}})^4$.
68. $(x^{\frac{1}{3}})^{-3}$.
69. $(a^{-6})^{\frac{1}{3}}$.
70. $(x^{-a})^0$.
71. $(x^0)^{-\frac{1}{2}}$.
72. $\dfrac{\sqrt{a}}{\sqrt[4]{a}}$.
73. $\dfrac{\sqrt[3]{x}}{\sqrt{x}}$.
74. $\dfrac{\sqrt[3]{4}}{\sqrt{2}}$.
75. $\dfrac{8^{\frac{4}{3}} x}{(-8)^{\frac{2}{3}} x^{\frac{1}{2}}}$.
76. $\dfrac{\sqrt[4]{x^6 y^6}}{\sqrt[3]{x} \cdot \sqrt{y}}$.

Exercises. *Review*

In each of the following exercises perform the indicated operations, and simplify each result:

1. $\dfrac{9\sqrt{20}-\sqrt{45}}{3\sqrt{5}}$.

2. $\dfrac{10\sqrt[3]{128}+2\sqrt[3]{16}}{4\sqrt[3]{2}}$.

3. $\dfrac{1+\sqrt{7}}{6-2\sqrt{7}}$.

4. $\dfrac{\sqrt{48\,y^3}-\sqrt{75\,y^3}+\sqrt{12\,x^2y}}{2x-y}$.

5. $\dfrac{1}{1-\sqrt{a}}-\dfrac{\sqrt{a}}{1+\sqrt{a}}$.

6. $\left(\dfrac{\sqrt{x}}{\sqrt{y}}-\dfrac{\sqrt{y}}{\sqrt{x}}\right)\dfrac{\sqrt{x^3y^2}}{\sqrt{xy}}$.

7. $(3\sqrt{7}-\sqrt{6}+2\sqrt{5})(3\sqrt{7}+\sqrt{6}-2\sqrt{5})$.

8. $(\sqrt{a+b}-\sqrt{a-b})\sqrt{a+b}$.

9. $(\sqrt[3]{a}+2\sqrt[3]{b})(\sqrt[3]{a^2}-2\sqrt[3]{ab}+4\sqrt[3]{b^2})$.

10. $\sqrt{4\,x^{2n}-24\,x^n+36}$.

11. $\sqrt[7]{3^6+3^6+3^6}$.

12. $\sqrt[5]{\dfrac{2^{x+3}}{8^x}\cdot\dfrac{4^{-x}}{4^{-1}}}$.

13. $\sqrt[r]{\dfrac{3^{4r}+9^{2r}}{9^r+3^{2r}}}$.

14. $\sqrt[n]{\dfrac{5^{n+2}-5^n}{24\,x^n}}$.

15. $2\sqrt{n}(m-n)^{-1}+2(\sqrt{m}+\sqrt{n})^{-1}$.

16. $(a^2+1)^{\frac{1}{2}}(a^2-1)^{-\frac{1}{2}}-(a^2-1)^{\frac{1}{2}}(a^2+1)^{-\frac{1}{2}}$.

17. $\dfrac{x(1-x)^{-\frac{1}{2}}+(1-x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$.

18. $\dfrac{(n-2)^{\frac{1}{2}}-3(n-2)^{-\frac{1}{2}}}{n-5}$.

19. $\dfrac{\sqrt{b^3c\sqrt[3]{a^5}}}{\sqrt[3]{a\sqrt{b^3c^3}}}\cdot\dfrac{\sqrt{a}}{b^2}$.

20. $\left(\dfrac{y^{-\frac{2}{3}}}{x^{-\frac{2}{3}}}-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)\left(\dfrac{xy}{1-x^{-1}y}\right)$.

21. $\left(a^{\frac{1}{3}}b^{-\frac{1}{2}}\sqrt{a^{-\frac{1}{3}}b\sqrt[4]{a^{\frac{4}{3}}}}\right)^3$.

22. $\left(\dfrac{x^{-3}}{y^{-\frac{2}{3}}z}\right)\div\left(\dfrac{\sqrt{x^{-\frac{1}{3}}}\cdot\sqrt[4]{y^2}}{x^2z^{-1}}\right)^{-2}$.

5. Equations and Formulas containing Radicals

91. Irrational Equations. An *irrational* or *radical* equation in one unknown is an equation in which the unknown quantity appears in a radicand or has a fractional exponent.

Thus, $x^{\frac{1}{2}} = 4$, $\sqrt{x^2+1} = 7 + x$, $\sqrt[3]{x} - 2 = 3$ are irrational equations.

Equations of this type can usually be solved very easily by the application of the following rule of equality:

If two quantities are equal, their corresponding powers are equal.

Thus, if $a = b$, it follows that $a^2 = b^2$, $a^3 = b^3$, $a^n = b^n$, and so on.

Illustrative Example 1. Solve the equation $3\sqrt{2x} = 4$.

Solution.

(1)	$3\sqrt{2x} = 4.$	Given.
(2)	$9 \times 2x = 16.$	Squaring.
(3)	$x = \frac{8}{9}.$	D_{18}.

Check. L.S. $= 3\sqrt{\frac{16}{9}} = 3(\frac{4}{3}) = 4$;

R.S. $= 4$.

Hence $\frac{8}{9}$ is a root.

NOTE. It is important to remember that in the evaluation of radicals only the *principal root* of each radical is to be used.

Illustrative Example 2. Solve the equation

$$2 - \sqrt{x^2 - 5} = x - 3.$$

Solution.

(1)	$2 - \sqrt{x^2-5} = x - 3.$	Given.
(2)	$-\sqrt{x^2-5} = x - 5.$	Tr.
(3)	$x^2 - 5 = x^2 - 10x + 25.$	Squaring.
(4)	$10x = 30.$	Tr.
(5)	$x = 3.$	D_{10}.

Check. L.S. $= 2 - \sqrt{4} = 2 - 2 = 0$;

R.S. $= 3 - 3 = 0$.

Hence 3 is a root.

These methods of solving can also be applied to the case of *formulas* containing radicals or fractional powers.

Illustrative Example 3. Solve the formula $a = \sqrt{b} + c$ for b.

Solution.		
(1)	$a = \sqrt{b} + c.$	Given.
(2)	$\sqrt{b} = a - c.$	Tr.
(3)	$b = a^2 - 2ac + c^2.$	Squaring.

Check. Substitute this value of b in the given equation.

Exercises. *Radicals in Equations and Formulas*

Solve each of the following equations, and check each result:

1. $\sqrt{x} = 5.$
2. $\sqrt{2x} = 6.$
3. $\sqrt{x+6} = 7.$
4. $\sqrt{x-3} = 10.$
5. $6\sqrt{3x} = 21.$
6. $\sqrt{\frac{x}{2}} = 5.$
7. $\sqrt{\frac{3x}{7}} = 6.$
8. $4\sqrt{x} = 3.$
9. $\sqrt{3x-2} = 7.$
10. $\sqrt{y} - 2 = 10.$
11. $\frac{3}{4}\sqrt{x} = 2\frac{1}{2}.$
12. $\sqrt[3]{x+2} = -1.$
13. $\sqrt{\frac{5x}{3}} = 10.$
14. $\sqrt[3]{\frac{9}{x}} = 3.$
15. $3\sqrt{x+2} = 4.$
16. $\sqrt{x} + 3\sqrt{x} = 8.$
17. $8 = 5\sqrt{x} - 2.$
18. $\sqrt[3]{x+3} = 5.$
19. $2 + \sqrt{x} = 5.$
20. $\frac{6+2x}{\sqrt{x}} = 3\sqrt{x}.$
21. $\frac{5+\sqrt{x}}{5-\sqrt{x}} = 4.$
22. $\sqrt[3]{3x-1} = \sqrt[3]{5x-7}.$
23. $\sqrt{x^2-13} = x - 1.$
24. $\frac{\sqrt{x}+6}{\sqrt{x}+3} = \frac{\sqrt{x}+3}{\sqrt{x}+1}.$
25. $\sqrt{9x^2-11} + 1 = 3x.$
26. $10 - \sqrt{3x-2} = 6.$
27. $\frac{\sqrt{x}-3}{\sqrt{x}+1} = \frac{\sqrt{x}-2}{\sqrt{x}+6}.$

Transform each of the following formulas as indicated:

28. Solve $V = \sqrt{2\,gh}$ for h.

29. Solve $d = 2.5\sqrt{2\,lw}$ for w.

30. Solve $e = \sqrt[3]{V}$ for V.

31. Solve $V = \sqrt{\frac{2K}{m}}$ for K.

6. Imaginary Numbers

92. Meaning of Imaginary Numbers. We have seen that $\sqrt{2}$ means the number whose square is 2; that is, $(\sqrt{2})^2 = 2$. In general, we know that $\sqrt[n]{a}$ is a number such that $(\sqrt[n]{a})^n = a$. Hence $\sqrt{-2}$ must be interpreted to mean the number whose square is -2; that is, $(\sqrt{-2})^2 = -2$. In like manner, $\sqrt{-a}$ is a quantity whose square is $-a$.

That is, $$(\sqrt{-a})^2 = -a.$$

In other words, for the first time we have a result that seems to contradict the rule of signs according to which the square of either a positive or a negative number is always positive. To meet this situation, mathematicians found it necessary to admit the existence of a new class of numbers. They called them *imaginary numbers*, and since it took many years to make the meaning of these numbers entirely clear, this misleading name has remained in use to the present day.

An imaginary number is an indicated even root of a negative number.

Thus, $\sqrt{-4}$, $\sqrt{-3}$, $\sqrt[4]{-6}$, $\sqrt{-\frac{1}{2}}$ are imaginary numbers.

Historical Note. For a long time imaginary numbers were viewed with suspicion even by the greatest mathematicians. However, the work of such mathematicians as Cardan (1545), Bombelli (1572), Stevin (1585), and especially Wallis (1673) prepared the way for their general acceptance. The terms "real" and "imaginary," in the sense described above, are due to Descartes (1637). The letter i for $\sqrt{-1}$ was used by the great Swiss mathematician Leonhard Euler (1748). The word

"conjugate" for a pair of complex numbers, such as $a + bi$ and $a - bi$, was used by Cauchy (1821).

It was not until 1797 that a Norwegian surveyor, Caspar Wessel, succeeded in finding a satisfactory method of representing imaginaries graphically. Other investigators independently arrived at similar results, especially J. R. Argand (1806) of Geneva, and C. F. Gauss (1800) of Göttingen.

Imaginary numbers are used extensively today in many branches of mathematics and its numerous applications. The fact that these numbers were found of great value in the development of modern electrical theory illustrates their practical use. To Euler (1748) is due the wonderful formula $e^{ix} = \cos x + i \sin x$, in which $e = 2.718 \cdots$. He is also responsible for the famous discovery that $e^{i\pi} = -1$.

By means of imaginary and complex numbers it is possible to "solve completely" many types of equations, and to state important laws and results in *general* form. Complex numbers represent the final step in the development of the number system of algebra.

93. Summary. In this chapter you have learned

(1) The fundamental laws of exponents.

(2) How to use positive, negative, zero, and fractional exponents.

(3) How to find the square root of a number (*a*) from a graph; (*b*) from a table.

(4) The fundamental principle of radicals and the fundamental laws of radicals.

(5) How to use expressions containing radicals.

(6) How to solve irrational equations.

(7) How to transform and to evaluate formulas containing radicals.

(8) The meaning of imaginary numbers.

Test on Chapter VI

Find the numerical value of each of the following:

1. $(\frac{4}{9})^0$.
2. $(\frac{3}{2})^{-3}$.
3. $16^{\frac{3}{4}}$.
4. $(-\frac{1}{27})^{-\frac{2}{3}}$.
5. $(4^{1.5})^2$.
6. $\left[(-8)^{-\frac{5}{3}}\right]^{-1}$.
7. $\dfrac{2^{-1}}{2^{-1} - 3^{-1}}$.
8. $\dfrac{2^0 - 8^{-1}}{2 - 3(4)^{-1}}$.

Write in simplest form each of the following:

9. $(5\,a^2b^{-3}) \times (2\,a^0b^4)$.

10. $(-6\,a^{-1}b^4) \div (3\,a^{-3}b^2)$.

11. $(-2\,x^{\frac{1}{2}}y^{-1})^4$.

12. $\left(\dfrac{9\,a^{-2}b^0}{16\,x^{-6}}\right)^{\frac{1}{2}}$.

13. $4\sqrt{27\,x^3}$.

14. $3\sqrt[3]{32\,a^2}$.

15. $a\sqrt{-25\,x^2}$.

16. $\dfrac{4\,n}{\sqrt{8}}$.

17. Multiply: $(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{-1} + b^{-2})(a^{\frac{1}{3}} + b^{-1})$.

18. Combine: $\sqrt[3]{24} - \sqrt[4]{64} + \sqrt[6]{8} - \sqrt[3]{\frac{8}{9}}$.

19. Expand: $(2\sqrt{5} - 5\sqrt{2})^2$.

20. Rationalize the denominator of $\dfrac{\sqrt{5}+\sqrt{7}}{2\sqrt{5}-\sqrt{7}}$, and find the value of the resulting expression to the nearest 0.01.

21. The formula for the radius of a cone in terms of its height and its volume is $r = \sqrt{\dfrac{3V}{\pi h}}$. Find r when $V = 94.26$, $h = 10$, $\pi = 3.142$.

Exercises. *Review*

1. Show that

$$(x^2 + y^2)(M^2 + N^2) = (xM + yN)^2 + (xN - yM)^2.$$

2. In $x^3 - 6\,x^2 + 3\,x - 4$, replace x by $x - 2$. Expand and combine similar terms.

3. Without actual division, show that $a + 3$ is a factor of $a^3 - 7\,a + 6$.

4. Which value of x makes $7(x - 3) + 4$ equal to $5 - 2(x - 6)$?

5. Solve $\dfrac{12 - x}{4\,x - 6} = \dfrac{1}{2} - \dfrac{x - 5}{2\,x - 3}$.

Simplify each of the following fractions:

6. $\dfrac{x-\dfrac{4}{x}}{1-\dfrac{5}{x}+\dfrac{6}{x^2}}$.

7. $\dfrac{\dfrac{m^2}{n^2}+\dfrac{n}{m}}{\dfrac{m}{n}-1+\dfrac{n}{m}}$.

8. Combine: $\dfrac{6}{a-1}+\dfrac{9}{2-a}+\dfrac{5}{1-a}+\dfrac{8}{a-2}$.

9. Solve for x the equation $\dfrac{n}{n+1}=\dfrac{x}{n^3+1}$.

10. From $t=\pi\sqrt{\dfrac{l}{g}}$, find t if $\pi=3.14$, $l=3$, $g=32$.

11. Write in the usual arithmetical form the numerical value given in each of the following statements:

(1) Light travels at the rate of 669.6×10^6 mi. per hour.

(2) The diameter of a molecule of oxygen is 3.2×10^{-8} cm.

12. Given the formula $V=\sqrt{\dfrac{E}{d}}$. Solve for d in terms of V and E.

13. A baseball team played n games and lost l of them. What per cent of the games did it win?

14. The marked price of a radio set was \$75. What was its original cost to the dealer, if he allowed a discount of 10% on the marked price and still made a profit of 25% on the cost?

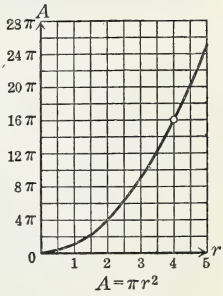
To discover *algebra* in the world, in life, in civilization, the test . . . is to look for . . . relations between measures of quantities. Do these occur? Look! Look!—E. R. Hedrick

CHAPTER VII

QUADRATIC EQUATIONS IN ONE UNKNOWN

1. Graphs of Quadratic Functions

94. Picturing a Formula involving the Square of One Variable. The formula for the area of a circle is $A = \pi r^2$. The diagram at the right shows the graph of this formula. One division of the vertical axis is chosen to represent 2π. Thus, for the value $r = 4$, $A = \pi 4^2 = 16\pi$.



Observe that the graph is *not* a straight line, but a curve. Hence, when such a graph is drawn freehand on ordinary squared paper, it is only *approximately* correct. Nevertheless, as in the case of straight-line graphs, it is possible to use such a graph to obtain *approximate* answers in solving two kinds of problems, within the range of values shown in the graph, as follows:

1. From any value of r we may obtain the corresponding value of A.

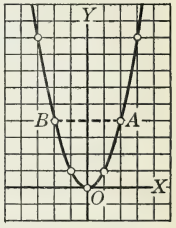
2. From any value of A we may obtain the corresponding value of r.

These considerations may readily be extended to the case of formulas which also call for *negative* values of the independent variable.

95. Functions of the Type x^2. By making a table such as the one shown below and plotting the points whose coordinates are the corresponding values of x and y, we are able to draw the graph of $y = x^2$.

$x =$	-3	-2	-1	0	1	2	3
$y = x^2 =$	9	4	1	0	1	4	9

This graph has the form of a curve which is called a *parabola*. The curve touches the x-axis at the origin; in other words, it passes through the point (0, 0).



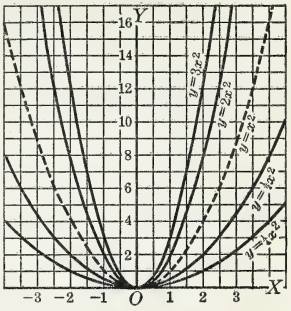
An inspection of the figure shows that for every point on the curve at the *right* of the y-axis, such as A, there is a corresponding point B on the curve at the *left* of the y-axis. That is, if the figure should be folded on the y-axis, the two branches of the curve which are on the left and the right side of the line of folding would coincide. We therefore say that the curve is *symmetric with respect to the y-axis*, and the y-axis is called the *axis of symmetry* of the curve, or simply the *axis* of the curve.

In like manner, the graph of $y = -x^2$ is a parabola which is situated *below* the x-axis. (Explain.) It is evident that the graph of $y = x^2$ and that of $y = -x^2$ are symmetrically placed with respect to the x-axis. (Why?)

96. Functions of the Type ax^2. Consider the following set of equations:

(1) $y = 2x^2$.
(2) $y = 3x^2$.
(3) $y = \frac{1}{2}x^2$.
(4) $y = \frac{1}{4}x^2$.

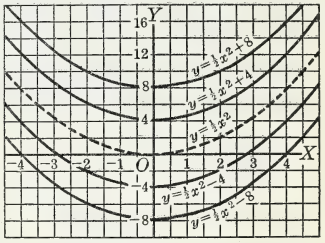
Observe that all the equations on page 198 are of the general form $y = ax^2$. The diagram shows that the graph of each of these equations is a parabola which touches the x-axis at the origin. It is evident that as the coefficient of x^2 increases, the two branches of the curve approach the y-axis, and as this coefficient decreases, the curve becomes flattened and approaches the x-axis. Again the y-axis is the *axis* of each of these parabolas.



In like manner, equations of the type $y = -ax^2$ are pictured by parabolas which are situated entirely *below* the x-axis. Evidently, for any given value of a, the graph of ax^2 and that of $-ax^2$ are symmetrically placed with respect to the x-axis.

97. Functions of the Type $ax^2 + c$. The diagram below shows the graph of each of the following four equations, which are all of the form $y = ax^2 + c$:

(1) $y = \frac{1}{2}x^2 + 4$.
(2) $y = \frac{1}{2}x^2 + 8$.
(3) $y = \frac{1}{2}x^2 - 4$.
(4) $y = \frac{1}{2}x^2 - 8$.



In each case it appears that the term c has the effect of "translating" the graph of $y = \frac{1}{2}x^2$ along the y-axis. The value of c indicates where the graph cuts the y-axis. It is the *y-intercept* of the curve. The y-axis is still the axis of symmetry of each curve.

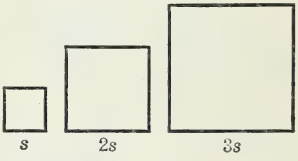
Exercises. *Plotting Graphs of Quadratic Functions*

1. The adjoining figure shows a circle inscribed in a square. If the diameter of the circle is d, write a formula for the area of the circle and one for the area of the square. Using the same axes, picture each formula.



2. The radii of three circles are r, $\frac{1}{2}r$, and $2r$ respectively. Using the same axes, draw the graphs which picture the areas of these circles.

3. Write the formula for the area (A) of each of the squares shown below. Using the same axes, draw the graphs which picture these formulas.



4. The formula for the surface of a sphere of radius r is $S = 4\pi r^2$. Draw a graph of this formula.

5. Using a table such as the one shown on page 198, draw the graph of $y = -x^2$.

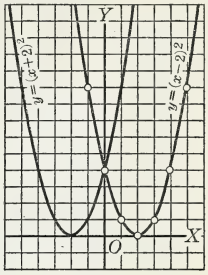
6. Using the same axes, draw the graph of each of the following equations:

(1) $y = -x^2 + 2$. (3) $y = -x^2 - 2$.
(2) $y = -x^2 + 4$. (4) $y = -x^2 - 4$.

Draw the graph of each of the following equations:

7. $y = 4x^2$.
8. $y = \frac{1}{4}x^2$.
9. $y = x^2 + 3$.
10. $y = x^2 - 8$.
11. $y = -3x^2$.
12. $y = -\frac{1}{2}x^2$.
13. $y = -x^2 + 2\frac{1}{2}$.
14. $y = 3x^2 - 1$.
15. $y = -2x^2 - 4$.

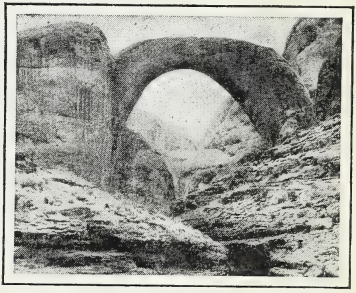
98. Functions of the Type $(x + a)^2$. Let it be required to draw the graphs of the equations (1) $y = (x - 2)^2$ and (2) $y = (x + 2)^2$. The graphs of these equations are shown in the adjoining figure.



Thus, the graph of $y = (x - 2)^2$ may be plotted with the aid of the following table:

$x =$	-1	0	1	2	3	4	5
$y = (x-2)^2 =$	9	4	1	0	1	4	9

Compare this table with the one used on page 198. Observe that the graph of $y = (x - 2)^2$ is exactly like that of $y = x^2$ except that it has been shifted two units to the *right* along the x-axis. That is, the new curve has an axis of symmetry, *parallel* to the y-axis, at the point $x = 2$. In like manner, by shifting the graph of $y = x^2$ two units to the *left* along the x-axis, we obtain the graph of $y = (x + 2)^2$. Similar conclusions hold with reference to the graph of any equation having the form $y = (x + a)^2$, where a may be either positive or negative.



Rainbow Natural Bridge, Utah

The curve of this natural arch of stone is an almost perfect parabola

99. Functions of the Type $ax^2 + bx + c$. The explanations which have been given in the preceding sections enable us

to sketch at once the graphs of many quadratic functions of the type $ax^2 + bx + c$ without the aid of a table.

In each case the graph of such a function is a parabola. When a is *positive*, the graph is turned *upward*; when a is *negative*, the graph is turned *downward*. Each of these parabolas has a vertical axis of symmetry which either coincides with the y-axis or is parallel to it. Observe also that when $x = 0$, we find that $y = c$. Hence the y-intercept of the graph is c.

NOTE. It will be found helpful in connection with the following exercises to draw the graph of $y = x^2$ on cardboard or stiff paper. The resulting parabola may then be cut out and used as a model.

Exercises. *Graphs of Quadratic Functions*

Without making a table, try to form a mental picture of the graph of each of the following equations, and then describe its position:

1. $y = (x-3)^2$.
2. $y = (x+4)^2$.
3. $y = (x-\frac{1}{2})^2$.
4. $y = 5(x+2)^2$.
5. $y = -(x+2)^2$.
6. $y = -3(x+2)^2$.
7. $y = (x+2)^2 - 3$.
8. $y = -(x+2)^2 + 5$.
9. $y = 2(x+2)^2 - 4$.
10. Describe the graph of $y = x^2 + 2x + 1$.

SUGGESTION. The equation may be written in the form $y = (x+1)^2$.

11. Describe the graph of $y = x^2 - 4x + 4$.
12. Describe the graph of $y = x^2 + 2x + 3$.

SUGGESTION. The equation may be written in the form

$$y = x^2 + 2x + 1 + 2 \quad \text{or} \quad y = (x+1)^2 + 2.$$

Sketch the graph of each of the following equations:

13. $y = x^2 + 4x + 5$.
14. $y = x^2 - 2x + 3$.
15. $y = x^2 + 6x + 10$.
16. $y = x^2 + 8x + 12$.
17. $y = -x^2 + 2x - 3$.
18. $y = -x^2 - 4x + 6$.
19. $y = 6(x+5)^2 - 2$.
20. $y = -4(x-6)^2 + 6$.

21. Describe the graph of $y = 2x^2 - 4x + 6$.

SUGGESTION. $y = 2x^2 - 4x + 6 = 2(x^2 - 2x + 3)$
$= 2(x^2 - 2x + 1 + 2) = 2[(x - 1)^2 + 2]$
$= 2(x - 1)^2 + 4.$

22. Sketch the graph of $y = 3x^2 + 12x + 7$.

23. Sketch the graph of $y = 2x^2 - 12x + 15$.

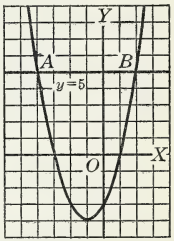
100. Solving Quadratic Equations Graphically. The fact that the graph of any function of the type $ax^2 + bx + c$ is a *parabola* is of importance in the graphic solution of quadratic equations, as the following examples will show:

Illustrative Example 1. Solve graphically the equation

$$x^2 + 2x - 3 = 0.$$

Solution. The left member of this equation is the quadratic function $x^2 + 2x - 3$. The graph of this function is shown in the figure below. It cuts the x-axis at two points, namely, at the points -3 and 1.

Now, at the points where the graph cuts the x-axis, $y = 0$. Hence at these points the equation $y = x^2 + 2x - 3$ becomes $0 = x^2 + 2x - 3$. That is, since the function $x^2 + 2x - 3$ is 0 for the values $x = -3$ and $x = 1$, it follows that -3 and 1 are the *roots* of the quadratic equation $x^2 + 2x - 3 = 0$.



Check. If $x = -3$, we have

L.S. $= 9 - 6 - 3 = 0$;
R.S. $= 0$.

If $x = 1$, we have

L.S $= 1 + 2 - 3 = 0$;
R.S. $= 0$.

We have therefore solved *graphically* the quadratic equation $x^2 + 2x - 3 = 0$.

NOTE. The points at which the graph cuts the x-axis may not represent *integral* values of x; that is, the roots may be *fractions* or *decimals*. In such a case the graph enables us to determine the roots at least *approximately*.

Illustrative Example 2. Solve graphically the equation

$$x^2 + 2x - 3 = 5.$$

Solution. The left member of this equation is the quadratic function $x^2 + 2x - 3$, thus giving us the parabola shown on the preceding page. We are to find the values of x which make this function equal to 5. That is, in this case the value of y is not 0, but 5. Hence we draw the graph of the equation $y = 5$. This line cuts the parabola in two points, A and B. The abscissas of these points are the required roots. (Why?) They are found to be 2 and -4. These values may be checked by substitution.

These illustrative examples suggest the following conclusions:

1. *Every quadratic equation in one unknown has two roots.*
2. *A quadratic equation of the type $ax^2 + bx + c = k$, where k is a constant, may be solved graphically by drawing the parabola which pictures the left member, and then drawing the graph of the line $y = k$. The abscissas of the intersection points of these graphs correspond to the roots of the given equation.*

Exercises. *Solving Quadratic Equations Graphically*

Draw the graph of $y = x^2 - 3x - 4$. From the graph determine the roots of the following equations:

1. $x^2 - 3x - 4 = 0$.
2. $x^2 - 3x - 4 = 2$.
3. $x^2 - 3x - 4 = -3$.
4. $x^2 - 3x - 5 = 0$.

Solve each of the following equations graphically, and check each root by substitution:

5. $x^2 + 3x - 4 = 0$.
6. $x^2 + 4x + 3 = 0$.
7. $x^2 + 3x - 10 = 0$.
8. $x^2 - 6x + 8 = 0$.

9. $x^2 - x - 6 = 0$.
10. $x^2 - 4x = 0$.
11. $x^2 + 5x = 0$.
12. $2x^2 + 3x - 3 = 6$.
13. $2x^2 - 3x - 1 = 1$.
14. $4x^2 - 9x + 5 = 0$.

Solve each of the following equations graphically, finding each root to the nearest tenth:

15. $x^2 - 2x - 2 = 0$.
16. $x^2 - 6x + 7 = 0$.
17. $x^2 + x - 1 = 0$.
18. $x^2 - 3x + 1 = 0$.
19. $2x^2 - 6x + 3 = 0$.
20. $4x^2 - 9x + 3 = 0$.

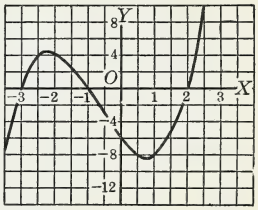
EXPLANATION. The graphic method used in solving quadratic equations can be extended to the case of equations which are of the first degree, or of a degree higher than the second.

For example, in order to solve the equation

$$x^3 + 2x^2 - 5x - 6 = 0$$

we draw the graph of the function

$$x^3 + 2x^2 - 5x - 6.$$



The resulting graph cuts the x-axis at the *three* points shown in the figure. That is, $x = -3$, $x = -1$, and $x = 2$ are the three roots of the given equation.

Solve each of the following equations graphically, and check each root:

21. $x^3 - 3x^2 - x + 3 = 0$.
22. $x^3 - 4x^2 + x + 6 = 0$.
23. $x^3 - 2x^2 - 5x + 6 = 0$.
24. $x^3 + 2x^2 - 8x = 0$.
25. $x^3 - 9x = 0$.
26. $x^3 + 6x^2 - x - 30 = 0$.
27. $x^4 - 4x^3 - 4x^2 + 16x = 0$.
28. $2x^4 - 5x^3 - 26x^2 + 41x + 60 = 0$.

Draw the graph of $y = 2x + 1$. From the graph determine the roots of the following equations:

29. $2x + 1 = 3$.
30. $2x + 1 = 2$.
31. $2x + 1 = 0$.
32. $2x + 1 = -1$.
33. $2x + 1 = 7$.
34. $2x + 3 = 6$.
35. $2x - 1 = 3$.
36. $2x + \frac{1}{2} = 4$.
37. $2x + 3 = 0$.

2. Algebraic Solution of Quadratic Equations

101. Quadratic Equations. If a quadratic equation contains both the second power and the first power of the unknown, it is called a *complete* quadratic.

Thus, $x^2 - 5x + 6 = 0$ is a complete quadratic equation.

We have already studied the solution of quadratic equations by graphs. We shall now show how to find the roots of quadratic equations algebraically. Generally speaking, algebraic methods give more accurate results. Why?

102. Solving Quadratic Equations by the Square-Root Method. Certain quadratic equations may readily be solved either by the *method of factoring* (see page 108) or by the *square-root method*. In using the latter method we make use of another rule of equality, which may be stated as follows:

If the same root of both members of an equation is taken, the resulting roots are equal.

Illustrative Example. Solve the equation $3x^2 = 45$.

Solution.

(1)	$3x^2 = 45$.	Given.
(2)	$x^2 = 15$.	D_3.
(3)	$x = \pm\sqrt{15}$.	Why?

Since $\sqrt{15} = 3.873$, the roots, to the nearest 0.01, are ± 3.87.

Check. If $x = \sqrt{15}$, we have

L.S. $= 3(\sqrt{15})^2$

$= 3 \times 15 = 45$;

R.S. $= 45$.

If $x = -\sqrt{15}$, we have

L.S. $= 3(-\sqrt{15})^2$

$= 3 \times 15 = 45$;

R.S. $= 45$.

Exercises. *Quadratic Equations*

Solve each of the following equations:

1. $x^2 = 49$.
2. $x^2 = 45$.
3. $ax^2 = b$.
4. $\frac{x^2}{3} = 27$.
5. $2x^2 = 32$.
6. $4x^2 = 100$.
7. $x^2 - 1 = 8$.
8. $\frac{x^2 - 5}{11} = 4$.
9. $5x^2 + 2 = 32$.
10. $7x^2 - 3 = 67$.
11. $cx^2 - a^2 = b^2$.
12. $3x - \frac{1}{2} = \frac{x(18 - x)}{6}$.

Transform and evaluate, as indicated, the following formulas:

13. $V = \pi r^2 h$. (1) Solve for r. (2) Find r if $V = 2156$ and $h = 14$.
14. $a^2 + b^2 = c^2$. (1) Solve for a. (2) Find a if $c = 10$ and $b = 3.1$.
15. $s = \frac{1}{2} gt^2$. (1) Solve for t. (2) Find t if $g = 32$ and $s = 160$.
16. $l : g = t^2 : \pi^2$. (1) Solve for t. (2) Find t if $l = 5$ and $g = 32$.
17. $K = \frac{1}{2} mv^2$. (1) Solve for v. (2) Find v if $K = 2560$ and $m = 5$.
18. $F \quad \frac{mv^2}{r}$. (1) Solve for v. (2) Find v if $F = 75$, $m = 80$, and $r = 3$.

103. Solving Quadratic Equations by Factoring. Many complete quadratic equations, as well as higher equations, can readily be solved by using the method of factoring explained on page 108. The following exercises will enable you to test your understanding of this method.

Exercises. *Solution by Factoring*

Solve the following equations by factoring, and check the roots:

1. $x^2 + 6x + 8 = 0$.
2. $x^2 - 2x - 15 = 0$.
3. $n^2 + 5n = 6$.
4. $2x^2 - 7x + 3 = 0$.
5. $3y^2 + 16y - 12 = 0$.
6. $5x^2 - 13x = 6$.
7. $9r^2 + 9r - 4 = 0$.
8. $12n^2 + 11n = -2$.
9. $4x^2 + 8x - 4 = 1$.
10. $5x^2 - 10x + 10 = x^2 + 3x$.
11. $3y^2 - 6y = 0$.
12. $n^2 - 25 = 0$.
13. $9x^2 - 24x = -16$.
14. $x^3 - 8x^2 + 16x = 0$.
15. $r^4 - 10r^2 + 9 = 0$.
16. $x^3 + 6x^2 + 3x - 10 = 0$.
17. $4x^3 - 17x^2 + 9x + 18 = 0$.
18. $\frac{x^2}{6} - x + \frac{3}{2} = 0$.
19. $\frac{y^2}{8} + \frac{5}{12}y = \frac{1}{3}$.
20. $6x^2 + 5ax - 4a^2 = 0$.

104. Solving Quadratic Equations by Completing the Square. Another algebraic method for solving quadratic equations, which may be used *in all cases*, is based on the formula $(a + b)^2 = a^2 + 2ab + b^2$, as the following illustrative examples will show.

Illustrative Example 1. Solve the quadratic equation

$$x^2 + 10x + 25 = 64.$$

Solution. The left member of this equation may evidently be written in the form $(x + 5)^2$. Then we have

$$(x + 5)^2 = 64.$$

Taking the square root of both members, we have

$$x + 5 = \pm 8.$$

If $x + 5 = 8$, $x = 3$;

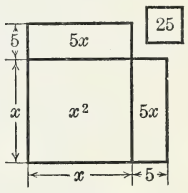
and if $x + 5 = -8$, $x = -13$.

Hence 3 and -13 are the required roots, which may be checked by substitution in the given equation.

Illustrative Example 2. Solve the equation

$$x^2 + 10\,x = 39.$$

Solution. The left member of this equation, namely, $x^2 + 10\,x$, may be pictured as shown in the adjoining diagram. The first term, x^2, is represented by the square whose side is x, while the second term, $10\,x$, represents the *sum* of the areas of the two equal rectangles, each of whose areas is $5\,x$. Since the length of each rectangle is x, the width must be 5, and hence the large square may be "completed" by the addition of 5^2, or 25. The resulting figure represents $x^2 + 10\,x + 25$, or $(x + 5)^2$.



We may therefore transform the left member of a quadratic equation having the form $x^2 + px = q$ into a *trinomial square*. *It is important to remember, however, that the quantity which is thus added to the left member of the equation must also be added to the right member.* (Why?) The work may then be completed as in the preceding example. Hence we have the following solution:

(1) $x^2 + 10\,x = 39.$	Given.
(2) $x^2 + 10\,x + 25 = 64.$	Completing the square.
(3) $x + 5 = \pm 8.$	Square root.
(4) If $x + 5 = 8$, $x = 3.$	S_5.
(5) If $x + 5 = -8$, $x = -13.$	S_5.

NOTE. The diagram above makes clear the following rule for finding the missing term of the trinomial square: *The missing term is the square of half the coefficient of* x. This rule applies only when the coefficient of x^2 is 1.

This method of solving a complete quadratic equation is called the *method of completing the square*. It may be summarized as follows:

1. *Reduce the given equation, if necessary, to the form* $x^2 + px = q$.

Thus, if $2\,x^2 + 8\,x = 12$, we have $x^2 + 4\,x = 6$, by division.

2. *Add to each member the square of half the coefficient of* x, *namely* $\left(\frac{p}{2}\right)^2$.

3. *Take the square root of each member, using the plus-or-minus sign* ($\pm$) *before the square root of the second member.*
4. *Solve the two resulting first-degree equations.*
5. *Check the resulting roots, each being expressed in its simplest form.*

Illustrative Example 3. Solve the equation

$$5x^2 - 11x - 12 = 0.$$

Solution.

(1)	$5x^2 - 11x - 12 = 0.$	Given.
(2)	$5x^2 - 11x = 12.$	A_{12}.
(3)	$x^2 - \frac{11}{5}x = \frac{12}{5}.$	D_5.
(4)	$x^2 - \frac{11}{5}x + (\frac{11}{10})^2 = \frac{12}{5} + \frac{121}{100} = \frac{361}{100}.$	$A_{(\frac{11}{10})^2}$.
(5)	$x - \frac{11}{10} = \pm\frac{19}{10}.$	Square root.
(6)	$x = \frac{11}{10} \pm \frac{19}{10}.$	$A_{\frac{11}{10}}$.

That is, $x = 3$ or $-\frac{4}{5}$.

Check. If $x = 3$, we have

$$\text{L.S.} = 5(9) - 11(3) - 12 = 45 - 33 - 12 = 0;$$
$$\text{R.S.} = 0.$$

If $x = -\frac{4}{5}$, we have

$$\text{L.S.} = 5(\tfrac{16}{25}) - 11(-\tfrac{4}{5}) - 12 = \tfrac{16}{5} + \tfrac{44}{5} - 12 = 0;$$
$$\text{R.S.} = 0.$$

Exercises. *Solving by Completing the Square*

In each of the following exercises supply the term which is needed to make the expression a trinomial square, and give the square root of the resulting trinomial:*

1. $x^2 + 4x + (\quad)$.
2. $x^2 + 8x + (\quad)$.
3. $n^2 - 2n + (\quad)$.
4. $x^2 - 3x + (\quad)$.
5. $y^2 + 7y + (\quad)$.
6. $n^2 + 15n + (\quad)$.
7. $a^2 - a + (\quad)$.
8. $x^2 + \frac{2}{5}x + (\quad)$.
9. $x^2 - \frac{1}{3}x + (\quad)$.
10. $x^2 + 2nx + (\quad)$.
11. $c^2 + 6ac + (\quad)$.
12. $y^2 - \frac{2}{3}ay + (\quad)$.

* A trinomial such as $x^2 + 10x + 25$ is sometimes called a *perfect* trinomial square (see page 97).

Solve each of the following equations by completing the square, and check the resulting roots. Express all irrational results in simplest radical form.

13. $x^2 + 4x = 5$.
14. $x^2 + 10x = 24$.
15. $x^2 - 6x - 8 = 19$.
16. $y^2 - 12y + 27 = 0$.
17. $n^2 + 8n + 6 = 0$.
18. $x^2 - 2x = 2$.
19. $r^2 - 5r + 6 = 0$.
20. $y^2 + 3y = 4$.
21. $x^2 - 3x = 6$.
22. $n^2 - 4n + 5 = 0$.
23. $x^2 + 6x - 3 = 0$.
24. $4x^2 - 8x = 5$.
25. $3y^2 - 5y + 2 = 0$.
26. $2n^2 - 10n - 1 = 0$.
27. $3x^2 - x - 3 = 0$.
28. $5r^2 + 2r - 1 = 0$.
29. $x + \frac{3}{2x} = \frac{11}{4}$.
30. $\frac{x+5}{3} = \frac{2}{x+3}$.
31. $\frac{60}{x+5} + 2 = \frac{60}{x}$.
32. $y^2 - 4cy = 5c^2$.
33. $x^2 + 5nx + 4n^2 = 0$.
34. $3x^2 - 5ax + 2a^2 = 0$.
35. $2x + \frac{1}{x} = 3$.
36. $\frac{7}{2x-1} + \frac{6}{x+2} = 2$.

Solve each of the following equations by completing the square, and give all irrational roots correct to the nearest 0.01:

37. $x^2 - 4x - 7 = 0$.
38. $x^2 + 5x + 2 = 0$.
39. $\frac{200}{x} + \frac{200}{x-3} = 10$.
40. $2y(y-6) = 15$.
41. $5x^2 + 8x - 21 = 0$.
42. $\frac{25}{r+4} + 2 = \frac{21}{r-4}$.
43. $3x^2 - 1.5x - 0.08 = 0$.
44. $(2n+3)^2 = n(10-n) + 15$.

45. The sum of the squares of two consecutive odd numbers is 394. Find the numbers.

46. Is it possible to find three consecutive whole numbers such that the sum of their squares is 194?

47. A business office which is 8 ft. by 10 ft. is to be enlarged by increasing each of its dimensions by an equal amount. If the floor space is to be doubled, what must be the new dimensions of the office?

105. Solving Quadratic Equations by Formula. The equation $ax^2 + bx + c = 0$, in which a, b, and c represent known numbers, is called the *standard*, or *general*, quadratic equation, since every quadratic equation may be written in that form. Hence the solution of this equation produces a general result, or *formula*, for finding the roots of *any* quadratic equation. This formula may be found as follows:

(1)	$ax^2 + bx + c = 0.$	Given.
(2)	$ax^2 + bx = -c.$	S_c.
(3)	$x^2 + \frac{b}{a} \cdot x = -\frac{c}{a}.$	D_a.
(4)	$x^2 + \frac{b}{a} \cdot x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$	$A_{\left(\frac{b}{2a}\right)^2}$.
(5)	$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$	Square root.
(6)	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$	$S_{\frac{b}{2a}}$.
That is,	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	

The resulting general solution, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is often called the *quadratic formula*. This formula is so

important that *it should be carefully memorized.* The following example will suggest how it may be used to solve *any* quadratic equation.

Illustrative Example. Solve the equation $9x^2 - 12x = 1$.

Solution. 1. $9x^2 - 12x - 1 = 0$. (Why?)

2. Comparing this equation with the standard form, $ax^2 + bx + c = 0$, we see that $a = 9, \quad b = -12, \quad \text{and} \quad c = -1.$

3. $b^2 - 4ac = 144 - 4(-9) = 144 + 36 = 180.$

The expression $b^2 - 4ac$ is called the *discriminant* of the equation (see page 228).

4. $x = \dfrac{-(-12) \pm \sqrt{180}}{18} = \dfrac{12 \pm 6\sqrt{5}}{18} = \dfrac{6(2 \pm \sqrt{5})}{18} = \frac{1}{3}(2 \pm \sqrt{5}).$

The roots may also be expressed in decimal form. That is, $x = 1.41$ or -0.08, to the nearest hundredth.

Check. The radical form of each root may be checked *exactly* by substitution, while the decimal values will check only *approximately.*

Exercises. *Solution by Formula*

Solve the following equations by using the quadratic formula:

1. $x^2 - 4x - 5 = 0$.
2. $x^2 + 6x + 8 = 0$.
3. $x^2 + x - 20 = 0$.
4. $x^2 - 10x + 24 = 0$.
5. $2x^2 - 11x - 6 = 0$.
6. $6x^2 + 5x = 6$.
7. $4n^2 + 35n + 24 = 0$.
8. $x^2 - 4x + 1 = 0$.
9. $4y^2 + 3 = 12y$.
10. $n^2 - 6n + 10 = 0$.
11. $\dfrac{n}{n-2} + \dfrac{n-1}{n^2-4} = \dfrac{3}{2}$.
12. $3x^2 - 5x + 4 = 0$.
13. $r^2 + 23 = 10r$.
14. $n^2 - 6n = 11$.
15. $25y^2 + 10y - 31 = 0$.
16. $10x^2 + 33x = -9$.
17. $x^2 + 2x + 21 = 0$.
18. $x^2 + 7ax - 8a^2 = 0$.
19. $3y^2 - 16cy + 5c^2 = 0$.
20. $x^2 - 8x + 16 - 9n = 0$.
21. $4x^2 + 4bx + b^2 = 12$.
22. $3 - \dfrac{3x-1}{6} = \dfrac{5}{3x-6}$.

Exercises. *Review*

Solve the following equations by the most convenient method, giving all irrational roots correct to the nearest 0.01:

1. $x^2 - 3x = 40$.
2. $3x^2 + 8x - 4 = 0$.
3. $x^2 - 10x + 15 = 0$.
4. $12y^2 - 5y = 3$.
5. $2n^2 - 4n + 1 = 0$.
6. $16x^2 + 24x = 11$.
7. $2y(y - 2) = 70$.
8. $2x^2 + 13x + 10 = 0$.
9. $15n^2 - 19n = -6$.
10. $y^2 + 8y + 14 = 0$.
11. $3w^2 - 6w + 2 = 0$.
12. $5y^2 - 24y = 5$.
13. $n^2 - (n - 2)^2 = (n - 4)^2$.
14. $n + \frac{5}{n} = 10\frac{1}{2}$.
15. $\frac{6}{x} + \frac{1}{3} = \frac{6}{x - 3}$.
16. $\frac{3x - 2}{3} - \frac{5(x^2 - 1)}{6x} = \frac{3}{2}$.
17. $\frac{5x}{x - 2} - \frac{4x - 5}{x + 2} = \frac{1}{2}$.
18. $w(2w + 3) = 104$.
19. $x^2 + 6ax + 8a^2 = 0$.
20. $3x^2 + 5bx = 2b^2$.
21. $2x^2 - 4ax + a^2 = 0$.
22. $x + \frac{1}{x} = a + \frac{1}{a}$.

106. Extraneous Roots. We have seen that an equation of the first degree in one unknown has one root and an equation of the second degree in one unknown has two roots. It may be shown that any process used in the solution of an equation which affects the degree of an equation also affects the number of roots.

Illustrative Example. Solve the equation $x + 2 = 4$.

Solution. The given equation is of the *first* degree. It has but *one* root, which is readily seen to be 2. Suppose, however, that we should decide to multiply both members by $x - 3$. By so doing, we should change the *degree* of the given equation, thus affecting the number of roots. The solution would then take the following form:

(1) $x+2=4.$	Given.
(2) $x^2-x-6=4x-12.$	$M_{(x-3)}$.
(3) $x^2-5x+6=0.$	Tr.
(4) $(x-2)(x-3)=0.$	Factoring.
(5) Hence $x=2$ or $x=3.$	Why?

The additional root 3, in the preceding example, does not satisfy the original equation. It is called an *extraneous root.* In general, a root which is *introduced* into an equation during the process of solving is called an *extraneous root.*

The solution of an *irrational* equation often leads to a quadratic equation, the roots of which may then be found by one of the methods discussed in the preceding pages. Each result must be carefully checked, however, to avoid the use of an *extraneous* root, which must be rejected.

Illustrative Example. Solve the equation

$$x+5-\sqrt{45-x^2}=8.$$

Solution. The equation may be written in the form

$$x-3=\sqrt{45-x^2}.$$

Squaring both members, we have

$$x^2-6x+9=45-x^2,$$

or

$$x^2-3x-18=0.$$

The roots of this equation are 6 and -3.

Check. If $x=6$, we have	If $x=-3$, we have
L.S. $=6+5-\sqrt{9}=11-3=8$;	L.S. $=-3+5-\sqrt{45-9}=2-6=-4$;
R.S. $=8$.	R.S. $=8$.
Hence 6 is a root.	-3 is an extraneous root.
	Hence -3 is *not* a root.

Exercises. *Irrational Equations*

Solve the following equations, rejecting extraneous roots:

1. $6=\sqrt{5x+1}.$
2. $4\sqrt{x+3}=7.$
3. $\sqrt{x^2-8}=x+4.$
4. $5-x=\sqrt{x^2+15}.$

5. $x+\sqrt{x+5}=7$.

6. $x-\sqrt{11-x}=-1$.

7. $y+7+\sqrt{3y+34}=19$.

8. $2y-1-\sqrt{3y+15}=7$.

9. $\sqrt{2x+4}+\sqrt{6x}=10$.

10. $x+2=\sqrt{x+1}+7$.

11. $x+3=\sqrt{x-6}+15$.

12. $\sqrt{s+4}+\sqrt{5s}=\dfrac{24}{\sqrt{s+4}}$.

13. $4y+\sqrt{y+6}=3y+14$.

14. $x+\sqrt{2x-4}=2(x-3)$.

15. $y-10=\sqrt{4y-3}+2$.

16. $\dfrac{3y+10}{\sqrt{y-3}}=5\sqrt{y-3}+\dfrac{1}{3}$.

Exercises. *Problems*

1. The sum of two whole numbers is 38. Their product is 105. Find the numbers.

2. The difference between two whole numbers is 6. Their product is 91. Find the numbers.

3. The product of two consecutive integers is 86 less than twice the square of the next consecutive integer. What are the three integers?

4. Is it possible to find three consecutive odd numbers such that the sum of the squares of the first two equals the square of the third?

5. Find a number such that when it is increased by its reciprocal the sum is 2.

6. A certain rectangle is 8 in. longer than it is wide. Its area is 560 sq. in. What are its dimensions?

7. The perimeter of a rectangle is 100 yd. Its area is 576 sq. yd. Find its dimensions.

8. If it requires 72 rd. of fence to inclose a rectangular 2-acre lot, what are its dimensions? (1 A. = 160 sq. rd.)

9. The base of a triangle exceeds its altitude by 2 in., and its area is 180 sq. in. Find the base and the altitude.

10. Can the sides of a right triangle ever be consecutive even integers? If so, find these integers.

11. An architect drew plans for a house in which the length of the foundation was to be 11 ft. more than the width. The estimated cost, however, proved to be too high. Hence the plans were changed by reducing the length by 6 ft. and increasing the width by 1 ft. This eliminated $13\frac{1}{3}\%$ from the area of the ground plan. What were the dimensions of the original foundation? of the new foundation?

12. A photograph 2 in. by 3 in. is to be enlarged to three times its original area. What will be the dimensions of the enlarged photograph, to the nearest hundredth of an inch?

SUGGESTION. The sides of the enlarged photograph have the same ratio as the corresponding sides of the original.

13. The enrolment of a certain high school is to be pictured by a square which is 12 in. on a side. If the enrolment of a second high school, which is one and one half times as large, is to be pictured on the same scale, how much larger must the side of this square be made?

*14. A dirigible made a continuous trip of 1950 mi. During the second half of the trip weather conditions improved and the airship was able to average 25 mi. per hour more than during the first half. If the entire trip required 22 hr. 45 min., what was the average rate during the first half of the trip? during the second half of the trip?

* Optional.

15. A department store bought a shipment of plain tumblers which cost \$24. Although 20 glasses were broken in the shipment, the store made a profit of \$14 by selling the remaining glasses for 4¢ apiece more than they cost. How many tumblers were ordered, and what did each cost?

16. A clothier bought a job lot of suits for \$800. He sold all but four of the suits, charging a uniform price for each suit and making a profit of \$12 on each. If he received \$1152 for all the suits he sold, what was the original cost of each suit?

17. Mr. Strong undertook the job of laying a cement walk. After working alone for 2 da., he hired Mr. Ade to assist him, and they finished the walk 4.8 da. later. If Mr. Ade had worked alone, he would have required 5 da. more than Mr. Strong to do the whole job. Find the time each man would have required to do the work alone.

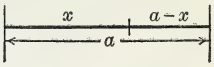
18. A gymnasium class in a large school is to be arranged in "solid square" formation. It is found that the class may be arranged either in one large square or in two smaller ones. When two squares are formed, the number of pupils in one side is one less and eight less, respectively, than the number in one side of the large square. How many pupils belong to this class?

*19. A rectangular kitchen floor which is 8 ft. by 11 ft. is to be covered with linoleum. When taking the measurements for the amount of linoleum needed to cover it, how much allowance may be made in measuring the length and the width to allow not more than 4 sq. ft. of waste?

* Optional.

20. The number of telephone connections (C) possible on a switchboard with which n telephones are connected is expressed by the formula $C = \frac{n(n-1)}{2}$. If 4950 connections are possible, how many telephones are there?

21. A line a is said to be divided "in extreme and mean ratio" when the larger part (x) of the line is such that $a : x = x : a - x$. Find x in terms of a.



22. The Greeks, who were lovers of the beautiful, discovered that rectangular objects are most pleasing in shape when the length (l) and the width (w) are so chosen that $\frac{w}{l} = \frac{l}{l+w}$. Solve for w in terms of l.

23. According to the relation stated in Exercise 22, what should be the width of a rectangular table top which is 6 ft. long?

*24. The distance s in feet that a body falls from rest in t seconds is given by the formula $s = \frac{1}{2} gt^2$, where $g = 32$, approximately. Taking the velocity of sound in air as 1150 ft. per second, find the depth of a shaft if a stone dropped into it is heard to strike the bottom in 4 sec.

25. The formula $h = vt - 16\,t^2$ is used to find the height in feet (h) which is reached by a body projected vertically upward with an initial velocity of v feet per second.

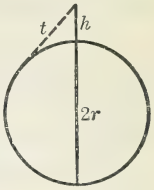
(1) Find t when $v = 100$ ft. and $h = 120$ ft.

(2) If $v = 80$ ft., after how many seconds will the body return to the ground?

26. What radius must be used to construct a circle whose area is to be half the area of a circle whose radius is 6 in.?

* Optional.

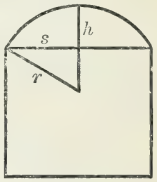
27. In the accompanying figure, h represents the height of a skyscraper, $2r$ the diameter of the earth, and t the distance that a person can see from the top of the skyscraper. It can be proved by geometry that $2r + h : t = t : h$.



(1) Find a formula for t.

(2) Since h is very small as compared with r, this formula may be written in the form $t = \sqrt{2rh}$. Find t in miles if $r = 4000$ mi. and $h = 1000$ ft. Show that t is approximately 39 mi.

28. The formula for the radius (r) of a circular arch is $r = \dfrac{s^2 + h^2}{2h}$, where s represents half the width of the span of the arch and h is the height of the arch.



(1) Solve for h in terms of s and r.

(2) Solve for h when $r = 5$ ft. and $s = 3$ ft.

107. Solving Equations in Quadratic Form. When an equation may be written so as to show the second power and the first power of the same unknown quantity, it is said to be an equation "in the quadratic form." Such an equation may be solved by any one of the methods which apply in the case of ordinary quadratic equations.

Illustrative Example 1. Solve the equation

$$x^4 - 11x^2 + 18 = 0.$$

Solution. Since x^4 is the second power of x^2, the equation may be written as a quadratic. To show this more clearly, we may substitute some letter such as u for x^2. Then we have

$$u^2 - 11u + 18 = 0.$$

Solving this equation for u, we have

$$u = 9 \quad \text{or} \quad u = 2.$$

That is, $$x^2 = 9 \quad \text{or} \quad x^2 = 2.$$

Hence $$x = \pm 3 \quad \text{or} \quad x = \pm \sqrt{2}.$$

Check. All four roots may be checked by substitution.

Illustrative Example 2. Solve the equation $2x^{\frac{2}{3}} + 5x^{\frac{1}{3}} = 3$.

Solution. Let $u = x^{\frac{1}{3}}$. Then the given equation may be written in the form $$2u^2 + 5u - 3 = 0.$$

Solving for u, we have

$$u = -3 \quad \text{or} \quad u = \tfrac{1}{2}.$$

That is, $$x^{\frac{1}{3}} = -3 \quad \text{or} \quad x^{\frac{1}{3}} = \tfrac{1}{2}.$$

Hence, solving these irrational equations (see page 191), we have

$$x = -27 \quad \text{or} \quad x = \tfrac{1}{8}.$$

Check. Each root may be checked by substitution.

Illustrative Example 3. Solve the equation

$$(x^2 + 2x)^2 - 11(x^2 + 2x) + 24 = 0.$$

Solution. Let $u = x^2 + 2x$. Then we have

(1)	$u^2 - 11u + 24 = 0.$	Substitution.
(2)	$(u - 8)(u - 3) = 0.$	Factoring.
(3)	Replace u by $x^2 + 2x$. Then we have	
	$(x^2 + 2x - 8)(x^2 + 2x - 3) = 0.$	Substitution.
(4)	$(x + 4)(x - 2)(x + 3)(x - 1) = 0.$	Factoring.
(5)	$x = -4$, $x = 2$, $x = -3$, or $x = 1$.	Why?

Check. All roots may be checked by substitution.

Illustrative Example 4. Solve the equation

$$x - 5 + 3\sqrt{x - 5} = 10.$$

Solution. Let $u = \sqrt{x - 5} = (x - 5)^{\frac{1}{2}}$. Then the original equation may be written in the form

$$u^2 + 3u = 10, \text{ or } u^2 + 3u - 10 = 0.$$

Factoring, we have $(u - 2)(u + 5) = 0$.

That is, $(\sqrt{x-5}-2)(\sqrt{x-5}+5)=0.$

Solving this equation, we find that

$$\sqrt{x-5}=2 \quad \text{or} \quad \sqrt{x-5}=-5.$$

Solving each of these equations for x, we have

$$x=9 \quad \text{or} \quad x=30.$$

Check. Determine by substitution whether or not both of these results satisfy the given equation.

Remember that *extraneous roots* (see page 215) must be rejected.

Exercises. *Equations in Quadratic Form*

Solve each of the following equations for all real roots:

1. $x^4-10x^2+9=0.$
2. $4x^4+35x^2=9.$
3. $x^6+7x^3-8=0.$
4. $2x^6+x^3-15=0.$
5. $x^{\frac{2}{3}}-5x^{\frac{1}{3}}+6=0.$
6. $3x^{\frac{2}{3}}+13x^{\frac{1}{3}}=10.$
7. $5x-7x^{\frac{1}{2}}-6=0.$
8. $x+2x^{\frac{1}{2}}=8.$
9. $6x^{-2}+7x^{-1}-20=0.$
10. $36x^{-4}-25x^{-2}+4=0.$
11. $x^{-6}-9x^{-3}+8=0.$
12. $4x^{-\frac{2}{3}}+7x^{-\frac{1}{3}}=2.$
13. $(x^2+5x)^2-2(x^2+5x)-24=0.$
14. $2(x-3)^4-11(x-3)^2+12=0.$
15. $(x^2-2x)^2-2x^2+4x=3.$
16. $(x+2)^{\frac{2}{3}}-6(x+2)^{\frac{1}{3}}+8=0.$
17. $(3x-1)+3(3x-1)^{\frac{1}{2}}=18.$
18. $\left(x+\frac{4}{x}\right)^2-\left(x+\frac{4}{x}\right)-20=0.$
19. $2\left(x-\frac{1}{x}\right)^2+7\left(x-\frac{1}{x}\right)=4.$
20. $2x+3+4(2x+3)^{\frac{1}{2}}-12=0.$
21. $(x^2-5x+7)^2+x^2-5x+5=0.$
22. $x^2-4x-8\sqrt{x^2-4x}+15=0.$
23. $x^2+3x+5\sqrt{x^2+3x}=14.$

3. Properties of Quadratic Equations

108. Relation between Roots and Coefficients. If we represent the roots of the quadratic equation $ax^2 + bx + c = 0$ by r_1 and r_2* respectively, we know from the *quadratic formula* (see page 212) that

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If we add these two equations, we have

$$r_1 + r_2 = -\frac{2b}{2a} = -\frac{b}{a}.$$

If we multiply the value of r_1 by that of r_2, we have

$$r_1 r_2 = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Now it is evident that the roots of $ax^2 + bx + c = 0$ are also those of $x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0$. (Why?)

Hence you will see that *when the coefficient of x^2 is 1*, the following relations exist:

1. *The sum of the roots is equal to the coefficient of x with its sign changed.*
2. *The product of the roots is equal to the constant term.*†

For example, if the given equation is $x^2 - 5x - 6 = 0$, the sum of the roots is 5, and the product of the roots is -6.

The relations stated above afford a very convenient method of testing the correctness of the roots of a quadratic equation.

* The symbols r_1 and r_2 may be read "r sub-one" and "r sub-two," or "the first root" and "the second root," respectively.

† In a quadratic equation such as $x^2 + 5x + 6 = 0$ the term 6 does not vary and hence is called the "constant term."

Illustrative Example. Show that $\frac{3+\sqrt{7}}{2}$ and $\frac{3-\sqrt{7}}{2}$ are the roots of the equation $2x^2 - 6x + 1 = 0$.

Solution. The given equation may be written in the form

$$x^2 - 3x + \tfrac{1}{2} = 0.$$

Now, $$r_1 + r_2 = \frac{3+\sqrt{7}}{2} + \frac{3-\sqrt{7}}{2} = \frac{6}{2} = 3.$$

Also, $$r_1 r_2 = \left(\frac{3+\sqrt{7}}{2}\right)\left(\frac{3-\sqrt{7}}{2}\right) = \frac{9-7}{4} = \frac{1}{2}.$$

Hence, since the values obtained for $r_1 + r_2$ and $r_1 r_2$ agree with the required relations of roots and coefficients, the given roots are correct.

Exercises. *Using the Relation between Roots and Coefficients*

In the case of each of the following equations, determine by inspection the sum and the product of the roots. Then solve the equation and test the correctness of your statements.

1. $x^2 + 7x - 9 = 0$.
2. $x^2 - 5x - 10 = 0$.
3. $2x^2 - 6x - 7 = 0$.
4. $3y^2 + 4y - 5 = 0$.
5. $8x^2 - 2x = 12$.
6. $3x + 9 = 5x^2$.
7. $9n^2 + 12n = -4$.
8. $1 - 4y + 4y^2 = 0$.
9. $2x^2 + 8x = 0$.
10. $16x^2 - 1 = 0$.

In the case of each of the following equations, test the correctness of the roots indicated in the parentheses:

11. $3x^2 + 13x - 10 = 0$. $(-5, +\frac{2}{3})$
12. $x^2 + 8x + 13 = 0$. $(4+\sqrt{3}, 4-\sqrt{3})$
13. $9y^2 - 12y = 1$. $\left(\frac{2+\sqrt{5}}{3}, \frac{2-\sqrt{5}}{3}\right)$
14. $2x^2 + 5x - 6 = 0$. $[\frac{1}{4}(-5+\sqrt{73}), \frac{1}{4}(-5-\sqrt{73})]$
15. $x^2 - \sqrt{3}x - 6 = 0$. $(2\sqrt{3}, -\sqrt{3})$

Determine the value of n in each of the following:

16. $x^2 + nx + 12 = 0$ if one root is to be -3.

SUGGESTION. Here $r_1 + r_2 = -n$ and $r_1 r_2 = 12$. But $r_1 = -3$. Hence $r_2 = -4$. (Why?) It follows that $r_1 + r_2 = -n = -7$.

17. $x^2 + nx + 18 = 0$ if one root is to be twice the other.

18. $y^2 - 6y + n = 0$ if one root is to be 4 larger than the other.

19. $x^2 + nx + 16 = 0$ if the two roots are to be equal.

20. $3x^2 - 10x = n$ if one root is to be 2 less than the other.

109. Formation of Equations. When we know the roots of a quadratic equation, the corresponding equation may readily be formed by either one of two methods.

FIRST METHOD. If the roots of the quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ are denoted by r_1 and r_2, the equation may at once be written in the form

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0. \quad \text{(Why?)}$$

Illustrative Example. Form the equation whose roots are -3 and $\frac{1}{4}$.

Solution. Since $r_1 + r_2 = -\frac{11}{4}$ and $r_1 r_2 = -\frac{3}{4}$, we have the equation

$$x^2 + \tfrac{11}{4}x - \tfrac{3}{4} = 0,$$

or

$$4x^2 + 11x - 3 = 0. \quad \text{(Explain.)}$$

SECOND METHOD. By reversing the method used in solving a quadratic equation by factoring, we may also obtain the equation when its roots are given.

Illustrative Example. Form the equation whose roots are -3 and $\frac{1}{4}$.

Solution. 1. If $x = -3$, then $x + 3 = 0$; and if $x = \frac{1}{4}$, then $x - \frac{1}{4} = 0$.

2. Hence $(x + 3)(x - \frac{1}{4}) = 0$, or $x^2 + \frac{11}{4}x - \frac{3}{4} = 0$.

3. That is, the required equation is $4x^2 + 11x - 3 = 0$.

NOTE. This method may readily be extended to the case of equations having *three* or more roots.

Exercises. *Formation of Equations*

Using the relation between roots and coefficients, write the quadratic equations whose roots are

1. $3, -4$.
2. $6, \frac{2}{3}$.
3. $5, -\frac{1}{4}$.
4. $-8, -2$.
5. $\frac{3}{4}, \frac{1}{3}$.
6. $-\frac{3}{8}, -\frac{2}{5}$.
7. $n, 3n$.
8. $-4k, \frac{3}{5}k$.
9. $2 + \sqrt{3}, 2 - \sqrt{3}$.
10. $-5 - 3\sqrt{2}, -5 + 3\sqrt{2}$.
11. $3 + \frac{1}{2}\sqrt{7}, 3 - \frac{1}{2}\sqrt{7}$.
12. $\frac{-2 + \sqrt{10}}{3}, \frac{-2 - \sqrt{10}}{3}$.
13. $6 + \sqrt{-1}, 6 - \sqrt{-1}$.
14. $\frac{1 + \sqrt{-5}}{2}, \frac{1 - \sqrt{-5}}{2}$.
15. $m - n, m + n$.
16. $\frac{a - b}{2}, \frac{2}{a - b}$.

By using the "factor method" explained above, write the equations whose roots are given below:

17. 5 and -2.
18. $\frac{3}{4}$ and $\frac{2}{3}$.
19. $\sqrt{3}$ and $-\sqrt{3}$.
20. $-b$ and $-4b$.
21. $5 + \sqrt{2}$ and $5 - \sqrt{2}$.
22. $-3 + 2\sqrt{5}$ and $-3 - 2\sqrt{5}$.
23. $\frac{3}{4} + \frac{1}{4}\sqrt{3}, \frac{3}{4} - \frac{1}{4}\sqrt{3}$.
24. $2 + \sqrt{7}, 2 - \sqrt{7}$.
25. $\frac{7 + \sqrt{-3}}{2}, \frac{7 - \sqrt{-3}}{2}$.
26. $c + d, c - d$.
27. $3, -2, 5$.
28. $4, -\frac{1}{2}, -3$.
29. $1, -3, 0$.

EXPLANATION. Any trinomial quadratic function of the type $ax^2 + bx + c$ can readily be factored by following these steps:

1. Put the given expression equal to zero, and solve the resulting equation, namely, $ax^2 + bx + c = 0$.

2. Using the roots thus obtained, write the equation

$$(x - r_1)(x - r_2) = 0.$$

3. The resulting equation is equivalent to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. (Why?) Hence we see that

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a(x - r_1)(x - r_2).$$

30. Factor the expression $6x^2 - 7x + 2$.

Solution. 1. If $6x^2 - 7x + 2 = 0$, we find that $x = \frac{1}{2}$ or $\frac{2}{3}$.

2. Hence $(x - \frac{1}{2})(x - \frac{2}{3}) = 0$, or $x^2 - \frac{7}{6}x + \frac{1}{3} = 0$.

3. That is, $6x^2 - 7x + 2 = 6(x^2 - \frac{7}{6}x + \frac{1}{3})$. (Explain.)

4. Hence $6x^2 - 7x + 2 = 6(x - \frac{1}{2})(x - \frac{2}{3}) = (2x - 1)(3x - 2)$. (In practice, steps 2 and 3 are omitted.)

Using the method explained above, factor the following:

31. $10x^2 - 7x + 1$.

32. $12x^2 - 17x + 6$.

33. $6x^2 + 5x - 4$.

34. $27x^2 + 51x - 56$.

35. $15x^2 + 52x - 84$.

36. $2x^2 - 10x + 5$.

37. $x^2 - 6x + 10$.

38. $x^2 + x + 1$.

110. Character of the Roots.* It is often helpful to be able to state, without solving a given quadratic equation, whether its roots will be *real* or *imaginary*. If the roots are *real*, we may wish to know whether they are *rational* or *irrational*. Finally, in certain problems we find it neces-

* It is suggested that the meaning of the terms "rational," "irrational," and "imaginary" be reviewed at this point. (See pages 174 and 193.) Throughout this discussion we shall assume that the coefficients of the quadratic equations to be considered are *real* and *rational* numbers.

sary to determine whether the roots are *equal* or *unequal.* All these questions concerning the "nature" of the roots of a quadratic equation can readily be answered by an inspection of the quadratic formula, or by using the general results $r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

1. *Are the Roots Real or Imaginary?* It is evident that if $b^2 - 4ac$ is *positive* or if it is *zero,* the roots are *real*; if it is *negative,* the roots are *imaginary.*

2. *Are the Roots Rational or Irrational?* If $b^2 - 4ac$ is a *perfect square,* or if it is *zero,* the roots are *rational*; if not, the roots are *irrational.*

3. *Are the Roots Equal or Unequal?* If $b^2 - 4ac$ equals *zero,* the roots are equal, for in that case the value of each root is $-\frac{b}{2a}$. (Explain.) If $b^2 - 4ac$ is *not* equal to zero, the roots are *unequal.*

The conclusions stated above are summarized in the following table:

$b^2 - 4ac$	Character of the Roots
Positive	Real, unequal, and (1) rational, or (2) irrational
Zero	Real, rational, equal
Negative	Imaginary

The expression $b^2 - 4ac$, which enables us to determine the character of the roots of a quadratic equation, is called the *discriminant* of the equation. It may be represented by the letter D.

Thus, if the equation is $x^2 - 4x + 3 = 0$, $D = b^2 - 4ac = 16 - 12 = 4$.

Illustrative Examples. In the case of each of the following equations determine the character of the roots without solving the equation:

1. $2x^2 + 5x - 12 = 0$.

Solution. $D = b^2 - 4ac = 25 + 96 = 121$.
Hence the roots are real, rational, and unequal.

2. $x^2 - 4x + 1 = 0$.

Solution. $D = b^2 - 4ac = 16 - 4 = 12$.
Since 12 is not a square, the roots are real, irrational, and unequal.

3. $4x^2 - 12x + 9 = 0$.

Solution. $D = b^2 - 4ac = 144 - 144 = 0$.
Hence the roots are real, rational, and equal.

4. $3x^2 + 2x + 1 = 0$.

Solution. $D = b^2 - 4ac = 4 - 12 = -8$.
Hence the roots are imaginary.

Exercises. *Character of the Roots*

By means of the discriminant determine the character of the roots in each of the following equations:

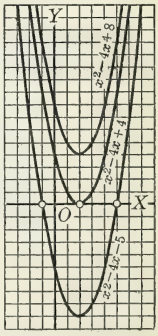
1. $x^2 + 9x + 8 = 0$.
2. $2x^2 - 7x = 30$.
3. $x^2 + x + 1 = 0$.
4. $y^2 - 10y + 25 = 0$.
5. $3x^2 - 5x = 1$.
6. $x^2 + 6x - 2 = 0$.
7. $4y^2 + 12y + 9 = 0$.
8. $\frac{x(x+3)}{2} = 35$.
9. $5y^2 + 2 = 3y$.
10. $6x^2 - 29x + 28 = 0$.
11. $y^2 - 12y = 8$.
12. $(2x + 3)^2 - 5 = 0$.
13. $2x(3x - 1) + 3 = 0$.
14. $x^2 - 25 = 0$.
15. $4x^2 = 9$.
16. $\frac{4}{x} + \frac{1}{2} = \frac{4}{x - 2}$.
17. $3x^2 - (x + 4)(2x - 7) = 70$.
18. $\frac{x - 12}{3x} - \frac{5(x - 5)}{9} + 3 = 0$.
19. $(3x + 7)(3x - 7) = -25$.

111. Variation of a Quadratic Function. Consider the following group of equations, together with their discriminants and their roots. Each equation is obtained by assigning zero value to the corresponding function.

	EQUATION	DISCRIMINANT	ROOTS
(1)	$x^2 - 4x - 5 = 0$	$D = 36$	$r_1 = -1;\ r_2 = 5$
(2)	$x^2 - 4x + 4 = 0$	$D = 0$	$r_1 = 2;\ r_2 = 2$
(3)	$x^2 - 4x + 8 = 0$	$D = -16$	$r_1 = 2 + 2\sqrt{-1}$ $r_2 = 2 - 2\sqrt{-1}$

If now we plot the graphs of the quadratic functions used in these equations, as shown in the diagram below, we shall be able to verify the correctness of the following statements:

1. *When the discriminant is* ***positive****, the graph of the function intersects the x-axis in two distinct points. That is, the roots of the given equation are* ***real*** *and* ***unequal.***
2. *When the discriminant is* ***zero****, the graph touches the x-axis at one point. In that case the roots are* ***real*** *and* ***equal.***
3. *When the discriminant is* ***negative****, the graph has no point in common with the x-axis. In such a case the roots are found to be* ***imaginary.***



The considerations presented above enable us to determine the value or values that a certain coefficient in a quadratic equation must have if the roots of the given equation are to be of a specified character.

Illustrative Example. Given the literal quadratic equation $x^2 + 2nx + 4 = 0$. Determine the value of n if the roots are to be (1) real and equal; (2) real and unequal; (3) imaginary.

Solution. 1. The roots are *real and equal* when $D = 0$. Hence we have $D = b^2 - 4ac = 4n^2 - 16 = 0$.

That is, $n^2 = 4$, and $n = \pm 2$.

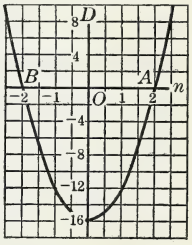
Using these values of n, we obtain the equations $x^2 + 4x + 4 = 0$ and $x^2 - 4x + 4 = 0$. It is readily seen that the roots of these equations are equal.

2. The roots are *real and unequal* when $D > 0$; that is, when $4n^2 - 16 > 0$.* This is the case when the absolute value of n is greater than 2.

3. The roots are *imaginary* when $D < 0$; that is, when $4n^2 - 16 < 0$. Hence the absolute value of n must be less than 2.

NOTE. The use of graphic methods is very helpful in a study of problems such as the preceding one.

Thus, if we plot the graph of the discriminant $D = 4n^2 - 16$, we obtain a parabola which cuts the horizontal axis at the points A and B as shown in the diagram at the right. It will be seen that these points correspond to the values $n = 2$ and $n = -2$, for which the required roots are *equal*. Values of n lying *between* A and B make the roots *imaginary*. (Why?) All other values of n make the roots *real* and *unequal*. (Why?)



Exercises. *Determining Literal Coefficients*

In each of the following equations determine the value of n which will make the roots equal:

1. $x^2 + 2nx + 16 = 0$.
2. $x^2 - nx + 25 = 0$.
3. $4x^2 - 7nx + 49 = 0$.
4. $9x^2 - 6nx + n = 0$.
5. $25x^2 + 20x + n^2 = 0$.
6. $n^2x^2 - 12x + 9 = 0$.

* The expression $D > 0$ means that "D is greater than 0," or *positive*. Likewise, $D < 0$ means that "D is less than zero," or *negative*.

7. $25\,x^2 + 5\,nx + 3\,n = 0$.

8. $36\,x^2 - 4\,nx + 25 = 0$.

9. $n^2x^2 - 40\,x + 100 = 0$.

10. $81\,x^2 - 108\,x + 4\,n^2 = 0$.

11. $x^2 - (n + 2)x + 16 = 0$.

12. $(n + 1)x^2 - 6\,x + n - 7 = 0$.

13. $(n + 2)x^2 - (3\,n - 2)x + 1 = 0$.

14. $(2\,n + 3)x^2 - 12\,x = 2\,n - 10$.

In the case of each of the following equations determine the values of n which will cause the roots to become (1) *real and equal;* (2) *real and unequal;* (3) *imaginary:*

15. $x^2 - 4\,nx + 64 = 0$.

16. $x^2 + 8\,x + n = 0$.

17. $nx^2 - 10\,x = 5$.

18. $2\,x^2 - nx + 8 = 0$.

19. $2\,nx^2 - 12\,x + 9 = 0$.

20. $2\,x^2 - 4\,x - n = 0$.

21. $3\,x^2 + nx + 3 = 0$.

22. $25\,x^2 - nx + 4 = 0$.

112. Summary. In this chapter you have learned

(1) How to represent quadratic functions graphically.

(2) How to solve a quadratic equation
  (*a*) By graph.
  (*b*) By square-root method.
  (*c*) By factoring (review).
  (*d*) By completing the square.
  (*e*) By the quadratic formula.

(3) How extraneous roots may be introduced.

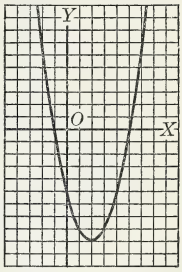
(4) How to solve equations in the quadratic form.

(5) How to use important properties of quadratic equations, namely:
  (*a*) The relation between roots and coefficients.
  (*b*) The relation of the discriminant to the character of the roots.

(6) How to solve problems involving quadratic functions.

Test on Chapter VII

1. The adjoining graph pictures the quadratic function $x^2 - 4x - 5$. From the graph determine the solutions of the following equations:



(1) $x^2 - 4x - 5 = 0$.
(2) $x^2 - 4x - 5 = 7$.
(3) $x^2 - 4x - 5 = -8$.
(4) $x^2 - 4x = 0$.

2. Solve the quadratic equations that follow, and check the roots by using the relation between roots and coefficients:

(1) $4x^2 + 5x - 6 = 0$. (2) $x^2 - 4x + 1 = 0$.

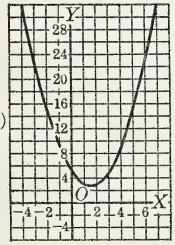
3. Form the quadratic equation whose roots are

(1) 3 and $-\frac{2}{5}$; (2) $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

4. Rearrange the following data so that all the items which belong together appear in the same column:

(1)

(*a*) $x^2 + 10x + 25 = 0$.

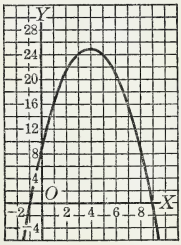


(*b*)

(*c*) Roots are real and unequal.

(2)

(*a*) $x^2 - 3x + 5 = 0$.

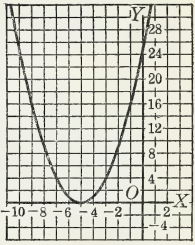


(*b*)

(*c*) Roots are real and equal.

(3)

(*a*) $-x^2 + 8x = -9$.



(*b*)

(*c*) Roots are imaginary.

5. If a ball is thrown vertically with an initial velocity of 88 ft. per second, the height which it will reach after t seconds is given by the formula $h = 88\,t - 16\,t^2$. After how many seconds will it return to the ground?

6. Given the function $x^2 - 6\,x + 5$.

(1) Draw the graph of the function from $x = -1$ to $x = 7$.

(2) On your graph indicate the axis of symmetry. Write its equation.

(3) From the graph determine the roots of the equation $x^2 - 6\,x + 5 = -4$.

Exercises. *Review*

1. Divide $6\,a^{-3} + 10\,a^{-2} - 4\,a^{-1} - 10 - 2\,a$ by $a^{-2} - 1$. Express the result with positive exponents.

2. By how much does $\dfrac{7}{x-3}$ exceed $\dfrac{7}{3-x}$?

3. Simplify $\dfrac{\dfrac{1}{x} - \dfrac{1}{y+z}}{\dfrac{1}{x} + \dfrac{1}{y+z}} \left(\dfrac{y^2 + z^2 - x^2}{2\,yz} + 1 \right)$.

4. Solve $\dfrac{x+3}{4\,x^2 - 1} + \dfrac{6\,x}{2(1-2\,x)} + \dfrac{3\,x+2}{2\,x+1} = 0$.

5. Prove by substitution that 4 is a root of the equation $x^{\frac{1}{2}} + 8\,x^{-\frac{1}{2}} = 6\,x^0$.

6. Express in decimal form 782×10^{-4}.

7. Mention two variables of which the interest on \$3000 is a function.

8. Find the value of $(0.25)^{-\frac{3}{2}} + \dfrac{52}{5 + 5^{-1}} + 10^0$.

9. Find the value of $\dfrac{(\sqrt{3+\sqrt{2}}+\sqrt{3-\sqrt{2}})^2}{6-2\sqrt{7}}$ correct to two decimal places.

Solve and check the following sets of equations:

10. $\dfrac{2x-y}{5}+3z=14,$
$x+y-z=10,$
$x-\dfrac{2y-z}{4}=6.$

11. $\dfrac{x}{a}+\dfrac{y}{b}=1,$
$y-z=2b,$
$\dfrac{x}{a}-\dfrac{z}{b}=3.$

12. Given the formula $s=\dfrac{n}{2}[2a+(n-1)d]$. Find the positive value of n if $s=140$, $a=5$, and $d=2$.

13. In an isosceles triangle each of the legs is l inches longer than the base. If the perimeter is P, find the base in terms of l and P.

Mathematical truths always have two sides or aspects. With the one they face and have contact with the world of outer realities lying in time and space. With the other they face and have relations with one another.—Sir T. Percy Nunn

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CHAPTER VIII

LOGARITHMS

1. Meaning and Use of Logarithms

113. Simplifying Computation by Logarithms. Many of the computations necessary in solving problems of engineering, surveying, astronomy, insurance, and the like would require an immense amount of labor and time if performed by the ordinary methods of arithmetic. Fortunately it is now possible to escape much of this drudgery. Early in the seventeenth century, John Napier of Merchiston, Scotland, pointed out how arithmetical calculations may be simplified by an ingenious use of exponents, or "logarithms."

This *method of logarithms* is based on the ordinary laws of exponents. It involves the use of special tables, *with the aid of which multiplication may be reduced to addition, division to subtraction, involution to multiplication, and evolution to division.* The use of this method requires but a small fraction of the effort usually required for all but the very simplest computations.

You will gain an idea of the power of the method of logarithms by studying the following examples. They are all based on the table below, which gives successive powers of 2 from the first power to the twelfth. (See page 188.)

$x =$	1	2	3	4	5	6	7	8	9	10	11	12
$y = 2^x =$	2	4	8	16	32	64	128	256	512	1024	2048	4096

Illustrative Example 1. Multiply 256 by 16.

Solution. $256 \times 16 = 2^8 \times 2^4 = 2^{12}$. From the table we see that $2^{12} = 4096$.

Illustrative Example 2. Divide 2048 by 128.

Solution. $2048 \div 128 = 2^{11} \div 2^7 = 2^4 = 16$.

Illustrative Example 3. Find the sixth root of 4096.

Solution. $\sqrt[6]{4096} = \sqrt[6]{2^{12}} = 2^2 = 4$.

Illustrative Example 4. Simplify $\dfrac{1024 \times 2048}{128^2 \times \sqrt[4]{256}}$.

Solution. $\dfrac{1024 \times 2048}{128^2 \times \sqrt[4]{256}} = \dfrac{2^{10} \times 2^{11}}{(2^7)^2 \times \sqrt[4]{2^8}} = \dfrac{2^{21}}{2^{14} \times 2^2} = 2^5 = 32.$

Evidently the ready use of this method depends on the completeness of the given table. Since all arithmetical computations are usually based on the number 10, and not on 2, we shall find it much more convenient to express the numbers to be combined as powers of 10. How this may be done, and how a corresponding table of logarithms may be used in simplifying computations, will be explained in the following pages.

114. Meaning of a Logarithm. In Chapter I the relation between involution, evolution, and logarithmation was explained (see pages 14–16). Thus, the statement $4^2 = 16$ suggests the following three problems:

(1) $4^2 = (\quad)$. [Involution]
(2) $(\quad)^2 = 16$. [Evolution]
(3) $4^{(\quad)} = 16$. [Logarithmation]

In the equality $4^2 = 16$ the exponent 2 is called the *logarithm* of 16 to the base 4. This relation is usually

written in the form $\log_4 16 = 2$. Similarly observe the following equivalent statements:

Exponential Form	Logarithmic Form
(1) $3^4 = 81$.	(1) $\log_3 81 = 4$.
(2) $10^2 = 100$.	(2) $\log_{10} 100 = 2$.
(3) $4^{-2} = \frac{1}{16}$.	(3) $\log_4 \frac{1}{16} = -2$.

Obviously any one of these exponential forms may at once be written in the corresponding logarithmic form, and conversely. In fact, *involution and logarithmation are inverse processes.*

In general, if $a^x = n$, we say that the exponent x *is the logarithm of* n *to the base* a, which may be written $\log_a n = x$. Hence we have the following definition:

The logarithm of a number **n** *to the base* **a** *is the exponent of the power to which the base* **a** *must be raised to produce* **n**.

Exercises. *Meaning and Use of Logarithms*

Make a table containing the first twenty powers of 2. Using this table, perform the following indicated operations:

1. 256×512.
2. 1024×64.
3. 4096×128.
4. $2048 \div 64$.
5. $65{,}536 \div 8192$.
6. $262{,}144 \div 16{,}384$.
7. 32^3.
8. 512^2.
9. $\sqrt[3]{262{,}144}$.
10. $131{,}072 \times 32{,}768 \div 8192$.
11. $524{,}288 \times 512^2 \div 131{,}072$.
12. $\dfrac{256^2 \times 64^3 \times 512}{1024^2 \times \sqrt[3]{32{,}768}}$.

13. $3.2 \times 6400 \div 512 \times 10.24$.

Suggestion. $3.2 = 32 \div 10 = 2^5 \div 10$ and $6400 = 64 \times 100 = 2^6 \times 10^2$.

Write in logarithmic form each of the following:

14. $3^2 = 9$.
15. $2^5 = 32$.
16. $2^{-1} = 0.5$.
17. $10^{-3} = 0.001$.
18. $x^2 = y$.
19. $b^c = d$.

Find the missing number in each case:

20. $\log_2 16 = \_\_\_\_$.
21. $\log_8 64 = \_\_\_\_$.
22. $\log_5 125 = \_\_\_\_$.
23. $\log_3 27 = \_\_\_\_$.
24. $\log_{10} 1000 = \_\_\_\_$.
25. $\log_{10} 0.01 = \_\_\_\_$.
26. $\log_{\frac{2}{3}} \frac{4}{9} = \_\_\_\_$.
27. $\log_4 \frac{1}{4} = \_\_\_\_$.
28. $\log_a a^2 = \_\_\_\_$.

State the logarithm of each of the following numbers, using the base specified:

29. 2, 1, $\frac{1}{2}$, 4, $\frac{1}{4}$, 8, 128; to the base 2.

30. 27, 3, 1, 243, $\frac{1}{9}$, $\frac{1}{81}$; to the base 3.

31. 10, 100, 1000, 10,000, 1,000,000; to the base 10.

32. 0.1, 0.01, 0.001, 0.0001, 0.0000001^2; to the base 10.

33. a, a^2, a^n, 1, $\sqrt{a}$, $\sqrt[3]{a}$, $a^{\frac{1}{n}}$; to the base a.

34. State the exponent, to the base 5, of each of the following numbers: 5, 1, 25, 125, $\frac{1}{5}$, $\frac{1}{25}$.

35. When the base is 4 and the logarithms are 1, 2, 3, 0, -1, -2, $\frac{1}{2}$, what are the numbers?

Verify each of the following statements:

36. $\log_2 8 + \log_2 4 - \log_2 32 = 0$.

37. $\log_{10} 10 + 3 \log_{10} 100 - 2 \log_{10} 1000 = 1$.

38. $\log_{10} \frac{1}{10} - 2 \log_{10} \frac{1}{100} + 3 \log_{10} \frac{1}{1000} = -6$.

39. $\log_6 6 + 5 \log_7 49 - 4 \log_2 8 = -1$.

40. $a \log_a a + a \log_a \frac{1}{a} + \log_a 1 = 0$.

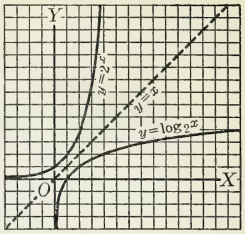
Find the value of x in each of the following cases:

41. $\log_{10} 10 = x$. 44. $\log_a x = b$.

42. $\log_b b = x$. 45. $\log_x 121 = 2$.

43. $\log_{10} x = 2$. 46. $\log_x n^3 = 3$.

47. Using the same axes, draw the graph of $y = 2^x$, and of $y = \log_2 x$.



SUGGESTION. Construct a table of corresponding values of x and y for each equation. Picture each of these tables. The resulting curves are shown in the figure at the right.

115. Common Logarithms. Any positive number except 1 may be used as the base of a table of logarithms. Because of our decimal system of numbers, however, 10 is the most convenient base for most purposes. Hence logarithms having 10 as their base are called *common logarithms*. Together they constitute the *common system* of logarithms.

HISTORICAL NOTE. Tables of common logarithms were first worked out by Napier's friend, Henry Briggs, an English professor of mathematics and science. In 1617 he published the decimal logarithms of the first thousand numbers. Napier's tables, published in 1614, involved a different base which is used extensively in higher mathematics.

Since the base 10 is used in all computations with logarithms in the common system, it is usually omitted in expressions written in the logarithmic form. Thus, $\log_{10} 1000 = 3$ is usually written in the form $\log 1000 = 3$.

116. Numbers as Powers of 10. To find the logarithm of a given number in the common system, we must answer the fundamental question, *Which power of 10 is equal to the given number?*

This question can readily be answered when the given number is a "perfect" power of 10. Thus, 10,000 is equal to 10^4. Hence $\log 10{,}000 = 4$. But suppose that any other positive number is to be expressed as a power of 10. For example, what is the value of log 72? Evidently, this question is equivalent to finding the missing number in the equation $10^{(\ )} = 72$. Now, since $10^1 = 10$, and $10^2 = 100$, the required exponent, or logarithm, lies *between* 1 and 2. By methods which cannot be explained at this point, it can be shown that $72 = 10^{1.8573}$, very nearly. Hence $\log 72 = 1.8573$. That is, the required exponent is 1 plus a *decimal* whose value can be determined to any required degree of accuracy.*

In like manner, observe the following illustrations of decimal powers of 10:

(1) $10^{1.1761} = 15$, and hence $\log 15 = 1.1761$;
(2) $10^{2.6675} = 465$, and hence $\log 465 = 2.6675$;
(3) $10^{3.9240} = 8394$, and hence $\log 8394 = 3.9240$.

117. Characteristic and Mantissa. From the preceding paragraph it is apparent that the logarithm of a given number, except in the case of "perfect" powers of 10, consists of an *integral* part (a whole number) and a *decimal* part.

The integral part of a logarithm is called its *characteristic*, and the decimal part is called its *mantissa*.

Thus, in the case of $\log 15 = 1.1761$, the characteristic is 1 and the mantissa is .1761. Similarly, in the case of $\log 8394 = 3.9240$, the characteristic is 3 and the mantissa is .9240.

* A more accurate value of log 72 is 1.857332. In general, such decimal exponents are *irrational* numbers. They are usually rounded off to 4 or 5 decimal places. Some decimal logarithms can be found easily by elementary methods. Thus, $\sqrt{10} = 10^{\frac{1}{2}} = 10^{0.5} = 3.1623$. Hence $\log 3.1623 = 0.5$. Similarly, since $31.623 = 10 \times 3.1623 = 10^1 \times 10^{0.5} = 10^{1.5}$, it follows that $\log 31.623 = 1.5$. Likewise, $\sqrt[4]{10} = 10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{10}} = 1.7783$. Hence $\log 1.7783 = 0.25$, and $\log 177.83 = 2.25$.

The characteristic of a logarithm, in the common system, can be found by *inspection*, while the mantissa is found from a *table*. A few simple but basic principles, which will now be discussed, are constantly used in determining the characteristic and the mantissa of any required common logarithm.

1. *Characteristic.* Observe the following table of powers of 10 and the corresponding logarithmic forms:

Powers of 10	Logarithms
$10^0 = 1$	$\log 1 = 0$
$10^1 = 10$	$\log 10 = 1$
$10^2 = 100$	$\log 100 = 2$
$10^3 = 1000$	$\log 1000 = 3$

Powers of 10	Logarithms
$10^0 = 1$	$\log 1 = 0$
$10^{-1} = 0.1$	$\log 0.1 = -1$
$10^{-2} = 0.01$	$\log 0.01 = -2$
$10^{-3} = 0.001$	$\log 0.001 = -3$

It is easily seen that the logarithm of a number between 1 and 10 lies between 0 and 1, and hence is 0 plus some decimal. If the number is between 10 and 100, its logarithm lies between 1 and 2, and hence is 1 plus some decimal, and so on. Similarly, the logarithm of a number between 1 and 0.1 lies between 0 and -1, and hence must be -1 plus some decimal. If the number is between 0.1 and 0.01, the logarithm is between -1 and -2, and hence must be -2 plus some decimal, and so on. It appears, then, that *the characteristic may be either positive or negative.* It is found convenient, however, *always to keep the mantissa positive.* Hence we have the following rules for determining the characteristic of a logarithm:

1. *The characteristic of the logarithm of a number greater than 1 is positive, and is one less than the number of integral places in the number.*

For example, $\log 97 = 1 +$ some mantissa,

and $\log 495.6 = 2 +$ some mantissa.

2. *The characteristic of the logarithm of a number between 0 and 1 is negative, and is numerically one more than the number of zeros immediately following the decimal point.*

For example, $\log 0.276 = -1 +$ some mantissa,

and $\log 0.0054 = -3 +$ some mantissa.

2. *Mantissa.* The following examples will explain the important principle underlying the use of a logarithmic table.

It can be shown that $8734 = 10^{3.9412}$. Hence it follows that $\log 8734 = 3.9412$. From this relation we can readily obtain the following statements:

Powers	Logarithms
$873.4 = \frac{8734}{10} = \frac{10^{3.9412}}{10^1} = 10^{2.9412}.$	$\log 873.4 = 2.9412.$
$87.34 = \frac{8734}{100} = \frac{10^{3.9412}}{10^2} = 10^{1.9412}.$	$\log 87.34 = 1.9412.$
$8.734 = \frac{8734}{1000} = \frac{10^{3.9412}}{10^3} = 10^{0.9412}.$	$\log 8.734 = 0.9412.$
$0.8734 = \frac{8734}{10{,}000} = \frac{10^{3.9412}}{10^4} = 10^{-1+9412}.$	$\log 0.8734 = -1 + .9412.$

The "mantissa law" suggested by these illustrations may be stated as follows:

Numbers which have the same figures in the same order and differ only in the position of the decimal point have the same mantissa in their logarithms.

118. Negative Characteristic. In dealing with a logarithm having a negative characteristic, it is necessary to remember that *the mantissa should always be kept positive.* Observe, for

example, that the logarithm of 0.0876, which is $-2+.9425$, is really a *binomial* equivalent to -1.0575. But in the value -1.0575 the decimal part, .0575, is also negative, which is not convenient for purposes of computation. Instead, we may use one of the following equivalent forms:

1. *The Monomial Form.* According to this plan, $-2+.9425$ is written in the form $\bar{2}.9425$. The minus sign *above* 2 indicates that the characteristic alone is negative. Similarly, $\log 0.479 = -1 + .6803 = \bar{1}.6803$.

2. *The Binomial Form.* The binomial $-2+.9425$ may be written in the form $10-2+.9425-10$, which is equivalent to $8+.9425-10$, or $8.9425-10$. In the resulting binomial, the mantissa remains positive. Observe that we have really added $10-10$, or 0, to the given logarithm, thus leaving it unchanged. In like manner, $\log 0.479 = -1 + .6803 = 9.6803 - 10$, and $\log 0.00294 = \bar{3}.4683 = 7.4683 - 10$.

Other equivalent binomial forms are also used. For example, $\log 0.00456 = \bar{3}.6590 = 0.6590 - 3 = 1.6590 - 4 = 2.6590 - 5$, and so on. In actual computations we may select the form which is found most convenient.

119. Using a Table of Common Logarithms. A table of logarithms usually contains only the mantissas, the decimal points being omitted. In using such a table, two problems arise, namely, (1) *to find the logarithm of a given number*, and (2) *to find the number corresponding to a given logarithm.* These problems will now be discussed, with reference to the four-place table on pages 312–313.

NOTE. A detailed discussion of *approximations* and *degrees of accuracy*, which will be helpful in the further study of logarithms, may be found in Mr. Betz's monograph on Direct Measurement, in the Third Yearbook of the National Council of Teachers of Mathematics.

1. *To find the Logarithm of a Given Number.* At present we shall consider only numbers containing not more than three significant figures.*

Illustrative Example 1. Find the logarithm of 34.8.

Solution. 1. The characteristic is 1. Hence we have

$$\log 34.8 = 1 + \text{some mantissa.}$$

2. The mantissa is the same as that of 348. (Why?)

In the table on page 312, we locate the number composed of the first two figures, namely **34**, in the column marked **N** (number). In the horizontal row containing 34 we locate the column headed by the *third* figure, **8**. Here we find the desired mantissa, namely, 5416. Thus we find that

$$\log 34.8 = 1.5416.$$

Illustrative Example 2. Find the logarithm of 0.00874.

Solution. Following along the row containing **87** to the column under **4**, we find that the desired mantissa is 9415. That is,

$$\log 0.00874 = -3 + .9415 = 7.9415 - 10.$$

2. *To find the Number corresponding to a Given Logarithm.* In this case we simply reverse the preceding process.

Illustrative Example. Find the number whose logarithm is 1.6415.

Solution. In the table on page 312 the mantissa 6415 is found in the horizontal row corresponding to **43** (in the **N** column) and in the column headed by **8**. Hence the succession of figures in the number to be found is 438. Since the characteristic is 1, the required number is 43.8.

The number corresponding to a given logarithm is called the *antilogarithm* of that number.

Thus, the antilogarithm of 1.6415 is 43.8, or antilog 1.6415 = 43.8.

For convenience in computation a table of antilogarithms is given on pages 314–315. The use of this table is explained on page 249.

* A decimal such as 0.0023 is said to have *two* significant figures, namely the combination 23. In such a case the zeros are *not* significant figures. Similarly 0.0004 has *one* significant figure, namely 4. A zero at the *end* of a number *may* be significant. Thus, if 40.602 is rounded off to 40.60, the number 40.60 has four significant figures. In a decimal such as 0.0023, the number 2 is called the *first* significant figure.

Exercises. *Using a Table of Common Logarithms*

Using the table on pages 312–313, verify each of the following logarithms:

1. $\log 25 = 1.3979$.
2. $\log 620 = 2.7924$.
3. $\log 0.047 = \bar{2}.6721$.
4. $\log 563 = 2.7505$.
5. $\log 0.00888 = \bar{3}.9484$.
6. $\log 0.205 = 9.3118 - 10$.

Find the logarithm of each of the following numbers:

7. 150.
8. 15.
9. 200.
10. 2.
11. 530.
12. 5.3.
13. 60.
14. 7.9.
15. 0.079.
16. 0.0083.
17. 0.9.
18. 115.
19. 354.
20. 49.8.
21. 6.72.
22. 0.0611.
23. 0.407.
24. 0.00183.
25. 0.000726.
26. 0.939.

Verify each of the following antilogarithms:

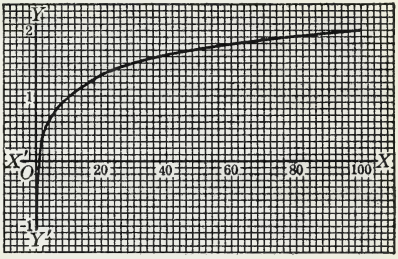
27. $\text{antilog } 2.2041 = 160$.
28. $\text{antilog } 1.6532 = 45$.
29. $\text{antilog } \bar{3}.7076 = 0.0051$.
30. $\text{antilog } 1.7716 = 59.1$.
31. $\text{antilog } 8.9112 - 10 = 0.0815$.
32. $\text{antilog } 9.7185 - 10 = 0.523$.

Find the numbers corresponding to the following logarithms:

33. 2.1461.
34. 1.1461.
35. 2.8129.
36. 0.8129.
37. 1.8633.
38. 2.6128.
39. $\bar{2}.6128$.
40. $\bar{3}.5798$.
41. $\bar{1}.9345$.
42. 0.7435.
43. 1.9058.
44. 0.9455.
45. 2.4997.
46. 0.4346.
47. $\bar{2}.1072$.
48. $8.7649 - 10$.
49. $7.9385 - 10$.
50. $9.9974 - 10$.
51. $9.7380 - 10$.
52. $8.0374 - 10$.

120. Interpolation. When a number contains four or more significant figures, the mantissa of its logarithm cannot be found directly from an ordinary four-place table. Hence a process of approximation or correction, called *interpolation,* is necessary. As a rule such a correction can be determined by a simple computation, which will now be explained.

The diagram represents the graph of $y = \log_{10} x$. Such a graph can readily be constructed with the aid of the table on pages 312–313.



If this graph is drawn on a large scale, it will be seen that as x increases, say beyond 10, the graph becomes very nearly a straight line. Hence *an increase in the value of x produces, very nearly, a proportional increase in the value of y.* This is the fact which underlies the process of interpolation.

Illustrative Example 1. Find the logarithm of 41.46.

Solution. 1. The required logarithm is 1 + some mantissa.

2. The mantissa of 41.46 is the same as that of 414.6. (Why?)

Since 414.6 lies between 414 and 415, the mantissa of log 414.6 is seen to lie between 6170 and 6180 (see table on page 312). The difference between 6180 and 6170 is 10. Hence the mantissa of log 414.6 is found by adding 0.6 of 10, or 6, to 6170, the mantissa of 414, thus making the required mantissa 6176.

That is, $$\log 41.46 = 1.6176.$$

NOTE. Tables of logarithms often give a column of differences. Observe that the last column of the tables on pages 312 and 313 is headed by D, which stands for "difference." Each number in these columns is the approximate difference between two successive mantissas appearing in the corresponding row.

Thus, in finding the logarithm of 4123, we find the mantissa 6149 in row 41 in column 2. Following along row 41 to column D we find 10, which is the difference between 6149 and the next larger (or the next smaller) mantissa. (In this particular case, the difference in a four-place table is not exactly 10, but if the mantissas were given to more decimal places, the difference would be found to be *approximately* 10.) We then add 0.3×10, or 3, to 6149, and thus find that $\log 4123 = 3.6152$.

Illustrative Example 2. Find the logarithm of 0.7562.

Solution. 1. The required logarithm is $-1 +$ some mantissa.

2. The required mantissa is the same as that of 756.2. (Why?)

Since 756.2 lies between 756 and 757, the required mantissa lies between 8785 and 8791. The difference between these mantissas is 6. Hence the mantissa of 756.2 is found by adding 0.2 of 6, or 1.2 (which, rounded off, is 1) * to 8785. That is,

$$\log 0.7562 = \bar{1}.8786 = 9.8786 - 10.$$

NOTE. The preceding examples show that in the case of *four-place* numbers the required correction can be obtained by using the formula $c = \frac{f}{10} \times d$, where c is the correction, f is the fourth figure in the given number, and d is the difference between the mantissas that are used in the process of interpolation. This difference is often referred to as the *tabular difference*.

Illustrative Example 3. Find the antilogarithm of 1.5614.

Solution. The table on page 312 shows that the mantissa 5614 lies between 5611 and 5623. These mantissas correspond to 364 and 365 respectively. The difference between 5623 and 5611 is 12, and the difference between 5614 (the given mantissa) and 5611 is 3. Hence the number corresponding to 5614 is found by adding $\frac{3}{12}$ of 1, or 0.25, to 364, thus making the number 364.25 (or 364.2, rounded off to four figures). But since the given characteristic is 1, we find that

$$\text{antilog } 1.5614 = 36.42.$$

NOTE. Five-place tables, which give the mantissas, to five decimal places, of the numbers from 1 to 9999, furnish directly the mantissas of four-place numbers.

* See page 61.

121. Using a Table of Antilogarithms. Instead of finding the antilogarithm of a number by the method explained in Illustrative Example 3, we may use a table of antilogarithms. Such a table contains the numbers corresponding to given logarithms. In the table on pages 314 and 315 the first two figures of the mantissa are found in the left-hand column (headed **M**) and the third figure is found in the top row. If the mantissa contains more than three significant figures, interpolation is necessary when using a four-place table.

Illustrative Example 1. Find the antilogarithm of 1.692.

Solution. In the table on page 315 we locate the mantissa of the first two figures, namely, 0.69, in the column marked **M** (mantissa). In the horizontal row containing 0.69 we locate the column headed by the figure 2. Here we find the number 4920. Since the characteristic of the logarithm is 1, we find that

$$\text{antilog } 1.692 = 49.2.$$

Illustrative Example 2. Find the antilogarithm of 2.5654.

Solution. In the table we locate the horizontal row containing the mantissa 0.56. Following along that row we find the number 3673 in the column headed by the third figure 5, and the number 3681 in the column headed by 6. Since 0.5654 lies between 0.565 and 0.566, the required number lies between 3673 and 3681. The difference between these numbers is 8 (as shown in the difference column *D*). Hence the antilogarithm of 0.5654 is found by adding 0.4 of 8, or 3 (approximately), to 3673, thus making the number 3676. Since the characteristic is 2, we find that

$$\text{antilog } 2.5654 = 367.6.$$

Check this result by finding the logarithm of 367.6.

A comparison of the method explained in Illustrative Example 3 on page 248 and that of the above examples shows that the use of a table of antilogarithms may save time in computation.

Exercises. *Using Interpolation*

With the aid of the process of interpolation explained above, find the logarithm of each of the following numbers:

1. 214.1.	7. 418.2.	13. 2.918.	19. 0.6007.
2. 142.1.	8. 296.2.	14. 4359.	20. 0.07748.
3. 438.1.	9. 875.4.	15. 56,350.	21. 0.0004809.
4. 453.5.	10. 94.45.	16. 35,725.	22. 0.90472.
5. 514.5.	11. 79.76.	17. 0.04573.	23. 0.0044383.
6. 731.5.	12. 2.177.	18. 0.2401.	24. 1.049.

Find the numbers corresponding to the following logarithms by using the process of interpolation:

25. 2.3434.	30. 1.9497.	35. 4.7787.	40. $9.0191 - 10$.
26. 2.6165.	31. 0.8801.	36. $\bar{1}.8328$.	41. $7.5181 - 10$.
27. 2.7608.	32. 0.4725.	37. $\bar{3}.7424$.	42. $8.8257 - 10$.
28. 3.8460.	33. 1.5451.	38. $\bar{2}.3516$.	43. 1.9819.
29. 3.9022.	34. 2.6598.	39. $\bar{4}.1585$.	44. $9.1313 - 10$.

122. The Fundamental Laws of Logarithms. Before we can use logarithms for purposes of computation, we must prove the following fundamental laws, which are all based on certain familiar rules of exponents.

1. *The logarithm of a product equals the sum of the logarithms of its factors.*

In symbols, $\log ab = \log a + \log b$.

Proof. 1. Let $a = 10^x$ and $b = 10^y$.

That is, $x = \log a$ and $y = \log b$.

2. Then $ab = 10^x \cdot 10^y = 10^{x+y}$. (Why?)

3. Hence $\log ab = x + y = \log a + \log b$.

2. *The logarithm of a quotient equals the difference between the logarithm of the dividend and the logarithm of the divisor.*

In symbols, $\log \frac{a}{b} = \log a - \log b.$

Proof. 1. If $a = 10^x$ and $b = 10^y$,

we have $x = \log a$ and $y = \log b.$

2. Then $\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y}.$ (Why?)

3. Hence $\log \frac{a}{b} = x - y = \log a - \log b.$

3. *The logarithm of a power of a number equals the logarithm of the number multiplied by the exponent of the power.*

In symbols, $\log a^n = n \log a.$

Proof. 1. Let $a = 10^x,$

whence $x = \log a.$

2. Then $a^n = (10^x)^n = 10^{nx}.$ (Why?)

3. Hence $\log a^n = nx = n \log a.$

4. *The logarithm of a root of a number equals the logarithm of the number divided by the index of the root.*

In symbols, $\log \sqrt[n]{a} = \frac{\log a}{n}.$

Proof. 1. If $a = 10^x,$

we have $x = \log a.$

2. Then $\sqrt[n]{a} = \sqrt[n]{10^x} = 10^{\frac{x}{n}}.$ (Why?)

3. Hence $\log \sqrt[n]{a} = \frac{x}{n} = \frac{\log a}{n}.$

2. Computations with Logarithms

123. Computations by Means of Logarithms. By applying the laws stated above it is possible to reduce multiplication to addition, and division to subtraction. Also, a power may be found by multiplication and a root by division. The method to be used in each case will be shown in the following examples.

Illustrative Example 1. Find the product of 6.73 and 42.5.

Solution. 1. THINK: $6.73 \times 42.5 = 10^x \times 10^y = 10^{x+y}$.

That is, $\log (6.73 \times 42.5) = \log 6.73 + \log 42.5$.

2. Hence we have the following computation:

$$\log 6.73 = 0.8280$$
$$\log 42.5 = \underline{1.6284}$$

Adding, we have $\log (\text{product}) = 2.4564$

Then $\text{antilog } 2.4564 = 286.1$.

3. Hence $6.73 \times 42.5 = 286$ (correct to three significant figures).

NOTE. In all computations it is very desirable to make a preliminary *estimate* of the result. Thus, in the preceding example we may estimate the product as 7×42, or 294.

By actual multiplication we find that $6.73 \times 42.5 = 286.025$. Results obtained by logarithms, as a rule, are only approximately correct. For most practical purposes, however, such approximations are sufficiently accurate.

Illustrative Example 2. Find the quotient of $816.2 \div 25.9$.

Solution. 1. THINK: $816.2 \div 25.9 = 10^x \div 10^y = 10^{x-y}$.

That is, $\log (816.2 \div 25.9) = \log 816.2 - \log 25.9$.

2. Hence we have the following computation:

$$\log 816.2 = 2.9118$$
$$\log 25.9 = \underline{1.4133}$$

Subtracting, we have $\log (\text{quotient}) = 1.4985$

Then $\text{antilog } 1.4985 = 31.52$.

3. Hence $816.2 \div 25.9 = 31.5$ (correct to three significant figures).

Illustrative Example 3. Find the value of 0.0532^3.

Solution. 1. THINK: $0.0532^3 = (10^x)^3 = 10^{3x}$.

That is, $\log 0.0532^3 = 3 \times \log 0.0532$.

2. Hence we have the following computation:

$$\log 0.0532^3 = 3 \times (8.7259 - 10).$$

Multiplying, we have $\log (\text{power}) = 26.1777 - 30$

$$= \bar{4}.1777.$$

Then $\text{antilog } \bar{4}.1777 = 0.0001505.$

3. Hence $0.0532^3 = 0.000150$ (correct to three significant figures).

Illustrative Example 4. Find the value of $\sqrt[4]{0.004963}$.

Solution. 1. THINK: $\sqrt[4]{0.004963} = \sqrt[4]{10^x} = 10^{\frac{x}{4}}$.

That is, $\log \sqrt[4]{0.004963} = \frac{1}{4} \log 0.004963$.

2. From the table we find that $\log 0.004963 = 7.6958 - 10$. To make division by 4 more convenient, we change the form of $7.6958 - 10$ into $37.6958 - 40$. Then we have

$$\log \sqrt[4]{0.004963} = \frac{37.6958 - 40}{4}.$$

Dividing, we have $\log (\text{root}) = 9.42395 - 10$

$$= \bar{1}.4240.$$

Then $\text{antilog } \bar{1}.4240 = 0.2655.$

3. That is, $\sqrt[4]{0.004963} = 0.2655.$

NOTE. Since 10^x is a *positive* number for all "real" values of x, a *negative* number does *not* have a "real" logarithm. Hence a difficulty arises when an expression whose value is to be found by logarithms contains *negative* factors. In such a case, we may *first find the absolute value of the expression* by logarithms and *then prefix the proper sign to the result.*

Thus, if the given expression is $34.9 \times (-76.8)$, we first find the value of (34.9×76.8) by logarithms, and then prefix the minus sign to the result.

Exercises. *Applying the Laws of Logarithms*

Find the value of each product by logarithms:

1. 16×52.
2. 3.25×88.
3. 87.5×144.
4. 576×0.625.
5. 2.56×0.125.
6. 187.5×12.8.
7. $(-305.8) \times 2.61$.
8. 42.8×217.3.
9. 2.59×31.8.
10. $(-88.2) \times 0.0745$.
11. $0.00563 \times (-791.4)$.
12. 0.6878×0.396.
13. 0.05149×0.2117.
14. 0.1234×0.0008.
15. $67{,}520 \times 0.01481$.

Find the value of each quotient by logarithms:

16. $640 \div 5.12$.
17. $93.6 \div 58.5$.
18. $296 \div (-4.21)$.
19. $492.5 \div 144$.
20. $70{,}850 \div (-39.8)$.
21. $0.9138 \div 0.0182$.
22. $0.05896 \div 0.0076$.
23. $984 \div 302.8$.
24. $(-37.8) \div 5.05$.
25. $0.2982 \div 0.08$.
26. $489.7 \div 16.23$.
27. $41.59 \div (-7.842)$.
28. $89.2 \div 177$.

SUGGESTION. Write the logarithm of the dividend in the binomial form. Thus,

$$\log 89.2 = 11.9504 - 10.$$

29. $58.4 \div 916$.
30. $338 \div 0.0612$.

Find the value of each power by logarithms:

31. 24^2.
32. 9.8^3.
33. 46.8^5.
34. 7.37^4.
35. 1.06^6.
36. 1.03^8.
37. 1.045^7.
38. 40.83^3.
39. 0.0786^2.
40. 0.00913^3.
41. $(-0.4687)^5$.
42. 0.8226^4.

Find the value of each root by logarithms:

43. $\sqrt{484}$.
44. $\sqrt[3]{-343}$.
45. $\sqrt[3]{9.84}$.
46. $\sqrt{4618}$.
47. $\sqrt[5]{7.343}$.
48. $\sqrt[3]{0.0837}$.
49. $\sqrt{0.005395}$.
50. $\sqrt[4]{0.1322}$.
51. $\sqrt[3]{0.03006}$.

124. Arrangement of Logarithmic Computations. In performing computations with logarithms, it is desirable to make a definite outline of the work to be done before a single logarithm is looked up in the table. A detailed plan for such work is shown below. A more condensed outline may, however, be found equally convenient.

Illustrative Example. Find the value of $\dfrac{7.520 \times 9.576}{3.942}$.

Solution. The computation is of the form $n = \dfrac{ab}{c}$.

Hence $$\log n = \log a + \log b - \log c.$$

Thus,* $$\frac{7.520 \times 9.576}{3.942} = \frac{10^x \times 10^y}{10^z} = \frac{10^{x+y}}{10^z} = 10^{x+y-z}.$$

NUMBERS	COMPUTATION
$a = 7.520$	$\log a =$ _____
$b = 9.576$	$\log b =$ _____
$c = 3.942$	$\log ab =$ _____
$n =$ ____	$\log c =$ _____
	$\log n =$ _____
	$n =$ _____

After the work has been outlined in this way, the logarithms are looked up in the table and the result is then readily determined.

NOTE. *Accuracy of Computations.* It should be remembered that a four-place table gives us decimals which, since they were rounded off, are only approximately correct. Hence *computations with a four-place table can give us results which are correct, at most, to four significant figures.* In case greater accuracy is desired, it is necessary to use a five-place, or more extensive, table.

* It is always helpful to rewrite the given expression in the "power form."

Since a four-place table gives results which, in general, are correct to $\frac{1}{2}$ of 1%, it is accurate enough for most kinds of practical problems. The numerical data which ordinarily arise in shops and laboratories are seldom determined beyond an accuracy of 1%. Five-place tables give results approximately correct to $\frac{1}{20}$ of 1%, and hence are sufficiently accurate for most engineering problems. Six-place tables are desirable in the case of computations involving compound interest, life insurance, annuities, and the like. The astronomer constantly uses a seven-place table, and even more extensive tables are needed for special purposes.

Exercises. *Computing with Logarithms*

Find the value of each of the following expressions:

1. $58.3 \times 3.47 \times 9.11$.
2. $7.28 \times 436 \times (-2.5)$.
3. $-83.2 \times 5.6 \times -6.75$.
4. $492.5 \times 0.028 \times 96.78$.
5. $1.062 \times 0.7934 \times 0.0009$.
6. 2560×1.04^{10}.
7. 3.142×2.25^2.
8. $51.84^3 \div 738$.
9. $9293 \div (-1.73)^5$.
10. $0.708^3 \times 64.08^4$.
11. $25 \times \sqrt{386.4}$.
12. $\sqrt[3]{81{,}200} \div \sqrt{97.3}$.
13. $\dfrac{5.89 \times 352.7}{-496.1}$.
14. $\dfrac{0.0718 \times 82.93}{725}$.
15. $\dfrac{0.188 \times (-53.26)}{0.06718}$.
16. $\dfrac{3.142}{6.4 \times 0.3333}$.
17. $\dfrac{72.48}{45.9 \times 38.7}$.
18. $\dfrac{0.4779 \times 0.073}{0.5624}$.
19. $\dfrac{99.46}{0.05071 \times 0.0008}$.
20. $\dfrac{78.21 \times 1.05}{3.87 \times 21.45}$.
21. $\dfrac{-643.6 \times 0.559}{84.43 \times (-0.075)}$.
22. $\dfrac{34.7^2 \times 10.95}{8.92^3}$.
23. $\dfrac{682 \times \sqrt{56.7}}{0.94^2 \times 377}$.
24. $\dfrac{\sqrt{60.84} \times \sqrt[3]{8076}}{1.414^4}$.

25. $\dfrac{32.77^2 \times \sqrt[3]{0.0761}}{\sqrt[5]{8975}}$.

26. $\dfrac{(-489)^{\frac{2}{3}} \times 0.078^{\frac{1}{2}}}{5005}$.

27. $\sqrt{\dfrac{31.48 \times 6789}{629.7 \times 2.46}}$.

28. $\sqrt[3]{\dfrac{54.3 \times 0.0784}{93.72^2}}$.

29. $\dfrac{1}{273}\sqrt{\dfrac{886 \times 26.5}{5300}}$.

30. $\dfrac{4\sqrt[3]{25 \times \sqrt{0.144}}}{\sqrt{50 \times 1.03^3}}$.

31. The volume of a cylinder is found by the formula $V = \pi r^2 h$. Find V when $r = 4.25$, $h = 13.65$, and $\pi = 3.142$.

32. The greatest height attained by a body projected vertically upward with an initial velocity of v feet per second is given by the formula $h = \dfrac{v^2}{2g}$. Find h if $v = 95$, $g = 32.2$.

33. The formula for the heat in calories developed by an electric current i flowing through a conductor having a resistance r during a time t is $H = \dfrac{ri^2t}{4.18}$. Find H when $r = 150$, $i = 0.75$, and $t = 45$.

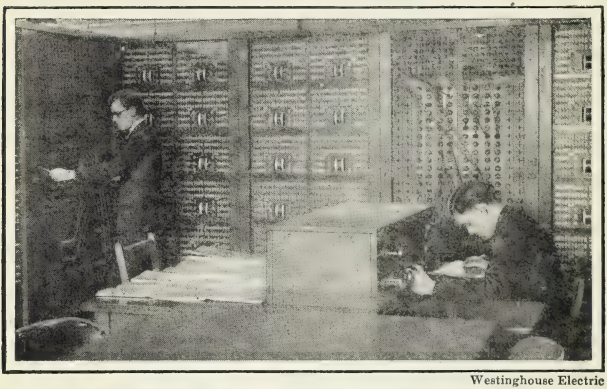
34. The elevation of the outer rail in a railway curve is given by the formula $e = \dfrac{GV^2}{32.2\,R}$. In this formula e is the elevation in feet; G, the gauge of the track in feet; V, the velocity of the train in feet per second; and R, the radius of the curve in feet. Find e when $G = 4$ ft. $8\frac{1}{2}$ in., $V = 45$ mi. per hour, and $R = 938$ ft.

35. The time in seconds for a complete oscillation of a simple pendulum of length l is given by the formula $t = 2\pi\sqrt{\dfrac{l}{g}}$. Find t when $l = 99.3$ cm., $g = 980$ cm., and $\pi = 3.142$.

36. The radius of a sphere of volume V may be found by the formula $r = \sqrt[3]{\dfrac{3V}{4\pi}}$. Find r when $V = 1574$ cu. in. and $\pi = 3.142$.

37. Find the number of acres in a field 65 rd. long and $47\frac{1}{2}$ rd. wide. (1 A. = 160 sq. rd.)

38. A floating body displaces its own weight of water. Find the weight in tons of a ship whose displacement is 75,000 cu. ft. (1 cu. ft. of water weighs 62.4 lb.)



Westinghouse Electric

A giant calculating machine

The calculating board is arranged on three sides of a square, each about 20 feet long. The instrument solves in a few days complicated problems in electricity which ordinarily require several weeks

39. The sun is approximately 93×10^6 mi. from the earth. If light travels at the rate of 186,000 mi. per second, how many minutes does it take light to come from the sun to the earth? How long does it take light to reach the earth from the nearest star, which is approximately 25×10^{12} mi. away?

40. If the number 5 is raised to the 50th power, how many digits are there in the resulting number?

3. Exponential Equations

125. Exponential Equations. An equation in which the unknown occurs in an exponent is called an *exponential equation.*

Thus, $2^x = 7$, $3^{x+1} = 9$, and $a^{cx} = b$ are exponential equations.

Exponential equations are of great importance. They arise, for example, in computations involving compound interest and in similar business problems.

Suppose that a sum of P dollars is put at compound interest for a period of years, interest being compounded annually and the rate of interest being 4 %. The following table shows the amount at the *end* of each year for a period of three years.

Year	Principal	Interest	Amount
1	P	$.04\,P$	$P + .04\,P = P(1.04)$
2	$P(1.04)$	$.04\,P(1.04)$	$*P(1.04) + .04\,P(1.04) = P(1.04)^2$
3	$P(1.04)^2$	$.04\,P(1.04)^2$	$P(1.04)^2 + .04\,P(1.04)^2 = P(1.04)^3$

In general, if the original principal is P and the rate is r, the amount at the end of n years is given by the formula

$$A = P(1 + r)^n.$$

If interest is compounded semiannually, $A = P\left(1 + \frac{r}{2}\right)^{2n}$; if quarterly, $A = P\left(1 + \frac{r}{4}\right)^{4n}$; and so on.

Observe that in the case of each of these formulas, if n is the unknown, we have an exponential equation.

The solution of simple types of exponential equations will become clear from the following examples.

Illustrative Example 1. Solve the equation $15^x = 296$.

Solution. 1. Taking logarithms of both members,† we have

$$\log 15^x = \log 296.$$

* Factoring the expression $P(1.04) + .04\,P(1.04)$, we have $P(1.04)(1 + .04)$, or $P(1.04)^2$.

† The logarithms of both members must, of course, be expressed in terms of the same base. Observe that we are really assuming the relation that if $a^x = a^y$, then $x = y$.

2. That is, $x \log 15 = \log 296$. (Why?)

3. Hence $$x = \frac{\log 296}{\log 15} = \frac{2.4713}{1.1761} = 2.101.$$

NOTE. Observe that here we are *dividing a logarithm by a logarithm.* Hence do not confuse the forms $\log \frac{296}{15}$ and $\frac{\log 296}{\log 15}$.

Illustrative Example 2. In how many years will a sum of \$1500 amount to \$2500 at 4% interest compounded annually?

Solution. 1. Using the compound-interest formula $A = P(1 + r)^n$, we see that $A = \$2500$, $r = 0.04$, and $P = \$1500$. Hence we have

$$2500 = 1500(1.04)^n.$$

2. Then $\log 2500 = \log 1500 + n \log 1.04$.

That is, $$n = \frac{\log 2500 - \log 1500}{\log 1.04}.$$

3. Hence $$n = \frac{3.3979 - 3.1761}{0.0170} = 13.05.$$

That is, the required number of years is approximately 13.

Illustrative Example 3. Solve the equation $9^{x-2} = 4.6^{2x+1}$.

Solution. 1. Taking logarithms of both members, we have

$$(x - 2) \log 9 = (2x + 1) \log 4.6.$$

2. That is, $(x - 2)0.9542 = (2x + 1)0.6628$.

(Then find the value of x by solving this linear equation.)

Exercises. *Exponential Equations; Compound Interest*

Find the value of x in each of the following equations without using a table:

1. $3^x = 9$.
2. $5^x = 125$.
3. $2^x = 64$.
4. $4^{-x} = \frac{1}{16}$.
5. $10^{x+1} = 100$.
6. $3^{2x+1} = 27$.
7. $2^{3x} = 512$.
8. $5^{x+3} = 625$.
9. $4^x = 8$.
10. $9^{2x} = 81$.
11. $0.25^x = 256$.
12. $100^{x^2} = 10{,}000$.

Solve each of the following exponential equations:

13. $5^x = 75$. 15. $5^{x+1} = 39$. 17. $3^{2x-1} = 7^{3x-2}$.

14. $10^{3x} = 71$. 16. $1.04^x = 2.66$. 18. $2.718^x = 10$.

NOTE. In the following examples the results found by means of a four-place table are only approximate. For more accurate computations, bankers use compound-interest tables.

If interest is compounded annually, find the amount of

19. \$600 for 5 yr. at 4%. 21. \$3200 for 6 yr. at $5\frac{1}{2}$%.

20. \$840 for 4 yr. at 5%. 22. \$4500 for 5 yr. at 6%.

If interest is compounded semiannually, find the amount of

23. \$940 for 3 yr. at 4%. 25. \$2000 for 5 yr. at 4%.

24. \$580 for 4 yr. at 6%. 26. \$5300 for 4 yr. at 6%.

4. The Slide Rule



126. The Slide Rule. The demand for greater speed and efficiency, so universal in this industrial and scientific age, has led to the invention of numerous devices for mechanical computation. Among these the *slide rule* is being used to an increasing extent by the engineer, statistician, business man, and scientist. The instrument in the illustration above is usually called the *Mannheim Slide Rule*. On such an instrument, multiplication, division, involution, evolution, etc. are accomplished by applying the principle of logarithms. Instead of tabulating the logarithms, the slide rule carries them in the form of scales or graduated lengths.

127. Summary. In this chapter you have learned

(1) The meaning of common logarithms.
(2) How to use tables of logarithms and antilogarithms.
(3) The fundamental laws of logarithms.
(4) How to compute with the aid of logarithms.
(5) How to solve simple exponential equations.
(6) How to calculate compound interest.

Test on Chapter VIII

In each of the following cases, find the value of x:

1. $\log_5 125 = x$. 2. $\log_3 x = 0$. 3. $\log_x 81 = 4$.

4. Show that $2 \log_2 2 + \frac{1}{2} \log_2 4 - 6 \log_4 2 = 0$.

5. Complete the following statement:

$$\log \frac{ab^2}{c} = \_\_\_\_\_ + \_\_\_\_\_ - \_\_\_\_\_.$$

By using logarithms find the value of the following:

6. 4.1×6.72. 8. $81.7 \div 2.91$. 10. $\sqrt[3]{98.74}$.

7. $\dfrac{0.56 \times 1.28}{6.42}$. 9. $\dfrac{76.4 \times 23.5}{0.27}$. 11. $\dfrac{0.061 \times 8.24^2}{7.5}$.

12. Which is greater, $(\frac{3}{2})^{20}$ or 1000?

13. At 4% interest, compounded annually, what is the amount of an investment of \$150 at the end of 20 yr.?

14. The number of gallons of water discharged per second from a pipe may be found approximately by using the formula $G = \sqrt{\dfrac{(15\ D)^5\ H}{L}}$. Find G if $D = 4$, $H = 50$, and $L = 700$.

Exercises. *Review*

Find the value of each of the following expressions, if $x = 8$:

1. $x^{\frac{2}{3}}$. 2. $\dfrac{(2x)^{\frac{3}{2}}}{12}$. 3. $\dfrac{3}{(x^{-1})^{\frac{4}{3}}}$. 4. $(x^{-\frac{1}{2}})^{\frac{2}{3}}$.

In each of the following exercises, write the equation whose roots have the given values:

5. $1 + \frac{1}{2}\sqrt{5}$, $1 - \frac{1}{2}\sqrt{5}$. 6. $\frac{3}{2} + \frac{1}{2}\sqrt{7}$, $\frac{3}{2} - \frac{1}{2}\sqrt{7}$.

7. (1) Draw the graph of $x^2 - 2x - 8$.

(2) For which values of x is the function (*a*) positive? (*b*) zero? (*c*) negative?

(3) What is the least value of the function?

(4) From the graph determine the roots of

(*a*) $x^2 - 2x - 8 = 0$. (*c*) $x^2 - 2x - 8 = -5$.

(*b*) $x^2 - 2x - 8 = -8$. (*d*) $x^2 - 2x = 15$.

8. A dairyman agreed to furnish milk which will contain 4% butter fat. How many gallons of cream testing 28% butter fat must he mix with milk testing 3% butter fat to obtain 100 gal. of the required mixture?

9. Given the formula $p = \dfrac{v}{608.3}\sqrt{\dfrac{w}{d}} - 0.14\,d$;

find p to the nearest 0.1 if $v = 2016$, $w = 1250$, and $d = 12\frac{1}{2}$.

10. Solve the following equation and check your solution:

$$\frac{\frac{1}{2}a - 3}{5} + \frac{1}{2}\left(\frac{3}{4}a - 10\right) = \frac{10 - a}{6} - \frac{4 - a}{4}.$$

Algebra is generous; she often gives more than is asked of her.
D'ALEMBERT

CHAPTER IX

SYSTEMS INVOLVING QUADRATIC EQUATIONS

1. Equations of the Second Degree in Two Variables: Conic Sections

128. The Conic Sections. In the case of any one of the common solids the question may arise as to what sort of plane figure or "section" is formed on a plane which passes through the solid.





Thus, every section of a *sphere* made by a plane is a *circle*. This fact is illustrated when an orange is cut into halves. Again, every section of a *pyramid* made by a plane parallel to the base is a polygon similar to the base.

Many centuries ago, certain Greek mathematicians began to study the sections which result when a circular *cone* is cut by a plane. They discovered that four different curves may be obtained, the form in each case depending on the relative position of the plane and the cone.



Indian tipi

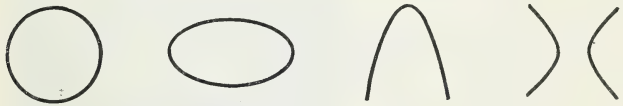
One of the earliest practical uses of the cone is illustrated by a conical shelter

The figures on page 265 suggest how these four curves, or *conic sections*, are related to a circular cone. If the cutting plane is parallel to the base of the cone, a *circle* results. If it is inclined at an angle, but still cuts all the elements* of the conical

* Any line connecting the vertex of a cone with any point in the perimeter of its base is called an *element* of the conical surface.

surface, the section is an *ellipse*. When the cutting plane is parallel to one of the elements of the conical surface, the curve of intersection is a *parabola*. Finally, if all the elements are extended beyond the vertex, as suggested in the fourth figure above, a "double cone" is formed. A cutting plane which does not pass through the vertex but intersects both parts of this double cone forms a curve of intersection which is called a *hyperbola*.



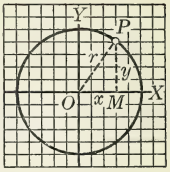


Circle Ellipse Parabola Hyperbola

HISTORICAL NOTE. The Greek mathematician Apollonius of Perga (about 225 B.C.) wrote a treatise on the conic sections which became one of the classic works of mathematics. The methods used by Apollonius and other Greek mathematicians were, however, purely *geometric* and hence difficult and laborious. In the 17th century it was shown by the distinguished French mathematicians Descartes and Fermat that curves such as the conic sections can be investigated much more easily by an *algebraic* approach.

At first the very great importance of these curves was not realized. A profound impression was created when Kepler (1571–1630 A.D.) established the theory that the path of the earth about the sun is an ellipse, and when, with the aid of conic sections, astronomers succeeded in giving a satisfactory explanation of the motions of the heavenly bodies.

It will be seen in the following pages that the four conic sections may be represented by certain typical equations, and we shall try to plot these curves with the aid of such equations.



129. The Circle. 1. *Definition.* A *circle* is a closed curved line in a plane, all points of which are equally distant from a fixed point in the plane, called the *centre.*

2. *Equation of a Circle.* Suppose that a circle is drawn on squared paper with its centre at the origin, as shown in the adjoining figure. Take any point P on this circle. Denote its coördinates, OM and MP, by x and y respectively. Then

$$\overline{OM}^2 + \overline{MP}^2 = \overline{OP}^2.$$

(Why?)

Hence, denoting OP by r, we have

$$x^2 + y^2 = r^2.$$

It is apparent that this equation is true for the coördinates of all other points on the circle. It is therefore called the *equation of a circle in which the origin is at the centre.*



The plan of this bridge involved the construction of a circular curve

3. *Plotting a Circle.* (*a*) Any equation of the type

$$x^2 + y^2 = r^2$$

will have as its graph a circle of radius r, the centre being the origin.

Thus, if the equation is $x^2 + y^2 = 25$, the value of r is 5. Again, if the equation is $2x^2 + 2y^2 = 12$, we find that $r = \sqrt{6}$. (Explain.)

(*b*) By writing the equation $x^2 + y^2 = r^2$ in the form $y = \pm\sqrt{r^2 - x^2}$, we may obtain a table of corresponding values of x and y and then plot the graph by means of this table.

DISCUSSION. Values of x larger than r would make y imaginary. (Explain.) Hence these values must be excluded.

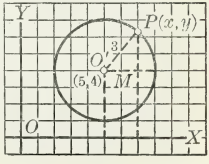
4. *Applications.* The circle is one of the "key" figures of geometry. Its properties help to establish many of the fundamental geometric constructions. The fact that countless objects have a circular form and that sections of the sphere and certain sections of cylinders and cones are circular is a sufficient indication of the very great importance of the circle.

Exercises. *The Circle*

Draw the graph of each of the following equations:

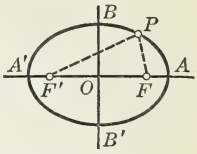
1. $x^2 + y^2 = 16$.
2. $x^2 + y^2 = 49$.
3. $x^2 + y^2 = 100$.
4. $x^2 + y^2 = \frac{1}{4}$.
5. $3x^2 + 3y^2 = 12$.
6. $5x^2 + 5y^2 = 35$.
7. $x^2 = 10 - y^2$.
8. $y^2 = 8 - x^2$.

9. If the centre O' of a circle has the coördinates $(5, 4)$, and its radius $O'P$ is 3, show that its equation is $(x-5)^2 + (y-4)^2 = 9$, or $x^2 + y^2 - 10x - 8y + 32 = 0$.



10. Show that the equation of a circle whose centre is at (a, b) and whose radius is r is $(x-a)^2 + (y-b)^2 = r^2$.

130. The Ellipse. 1. *Definition.* If a point P in a plane moves so that the *sum* of its distances from two fixed points in the plane, such as F and F' in the figure, is *constant*, it describes a curve which is called an *ellipse*.



The points F and F' are called the *foci* of the ellipse. Hence, if P moves so that $PF + PF'$ is always the same, it describes an ellipse.

An ellipse has two axes of symmetry, which are seen to be perpendicular to each other. The line-segment AA' on which the foci are located is called the *major axis*. Similarly, the line-segment BB' is called the *minor axis*. Evidently, AA' and BB' bisect each other at O. The point O is called the *centre* of the ellipse.

2. *Equation of an Ellipse.** If the centre of the ellipse is at the origin, and if the ellipse is symmetrically placed with reference to the x-axis and the y-axis, its equation has the form

$$ax^2 + by^2 = c,$$

where a, b, and c *agree* in sign.

3. *Plotting an Ellipse.* The method used in plotting an ellipse may be inferred from the following example.

Illustrative Example. Plot the graph of $4x^2 + 9y^2 = 36$.

Solution. 1. By making $y = 0$, we obtain the x-intercepts, ± 3. Similarly, the y-intercepts are ± 2. These values enable us to determine the axes of the ellipse.

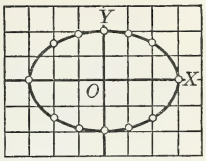
2. The given equation may be written in the form

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}.$$

* The equation of the ellipse, and of other conic sections, can readily be derived by methods similar to the one used in the case of the circle. The derivation of these equations is the task of a branch of mathematics called *analytic geometry*.

3. With the aid of this equation, we may readily obtain the following table of approximate corresponding values of x and y:

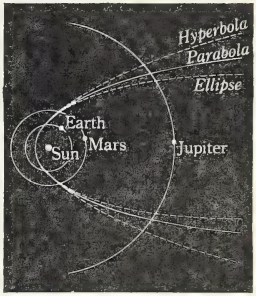
$x =$	0	± 1	± 2	± 3
$y =$	± 2	± 1.9	± 1.5	0



4. By plotting the points suggested by this table and drawing a smooth curve through them, we obtain an ellipse such as the one shown in the figure.

Discussion. Observe that when x equals either $+1$ or -1, $y = \pm 1.9$. Values of x numerically greater than 3 would make y imaginary. (Why?) Hence such values must be excluded.

4. *Applications.* The ellipse is a figure of great importance and interest. It is found very commonly in nature and it arises extensively in the arts and sciences. The orbit of each planet in the solar system is an ellipse, the sun being at one focus. Architects and engineers frequently employ the ellipse in the construction of arches, bridges, and auditoriums. In a "whispering gallery," sounds which originate at one focus are reflected to the other. Elliptical gears, which are used in machines such as planers, punches, and hay-presses, provide a combination of both slow and steady motion with a quick return motion. "Ovals" in the form of ellipses are widely used in applied design. Under certain conditions the shadow of a circular or spherical object is elliptical in shape.



Orbits of planets

The orbits of the planets are elliptical, although the orbit of the earth is almost circular. Most comets have parabolic orbits, although the orbits of some are distinctly elliptical and a few are hyperbolic

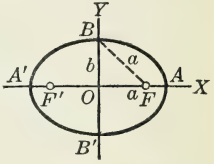
Exercises. *The Ellipse*

Plot the graph of each of the following equations:

1. $x^2 + 4y^2 = 16$.
2. $9x^2 + 4y^2 = 36$.
3. $4x^2 + 25y^2 = 100$.
4. $16x^2 + 9y^2 = 144$.
5. $25x^2 + 16y^2 = 400$.
6. $9x^2 + 36y^2 = 324$.
7. $4x^2 + y^2 = 20$.
8. $3x^2 + 4y^2 = 24$.
9. $0.36y^2 + 0.25x^2 = 9$
10. $9x^2 + 4y^2 = 0$.

EXPLANATION. When an ellipse is placed as shown in the figure, OA and OB are called the *semi-axes*. In that case, if $OA = a$ and $OB = b$, it can be shown that the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{or} \quad b^2x^2 + a^2y^2 = a^2b^2.$$



Again, if F and F' are the foci, it can be shown that $OF = \sqrt{a^2 - b^2}$. Also, using the same notation, the formula for the *area* of an ellipse is $A = \pi ab$. An approximate formula for the *perimeter* of an ellipse is $P = \pi\sqrt{2(a^2 + b^2)}$.

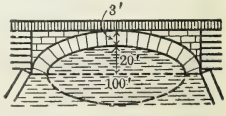
Using the notation explained above, write the equation of the ellipse if

11. $a = 5$, $b = 4$.
12. $a = 13$, $b = 12$.
13. $a = 10$, $b = 8$.
14. $a = 26$, $b = 5$.

15. An arch over a gate is to have the form of a semi-ellipse 60 in. long and 18 in. high, as shown in the diagram at the right. Write the equation of the ellipse. Plot the ellipse, using any convenient scale.

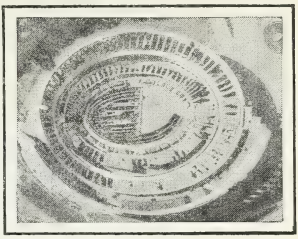


16. A horizontal road passes over a bridge which has a semi-elliptical arch. The roadway is 3 ft. above the vertex of the arch. If the span of the arch is 100 ft. and the height is 20 ft., how far is the roadway above the arch, at intervals of 10 ft.?



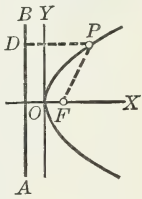
SUGGESTION. Let the y-axis pass through the vertex of the arch.

17. One of the most famous elliptical buildings ever built is the Colosseum in Rome. Its ground plan is an ellipse 615 ft. long and 510 ft. wide, and its arena is an ellipse 281 ft. by 177 ft. Write the equation of the ground plan and also of the arena. Find the area and the perimeter of the ground plan.



131. The Parabola. 1. *Definition.* When a point P moves so that its distance from a fixed line is always equal to its distance from a fixed point, it describes a *parabola.*

The fixed line, represented by AB in the figure, is called the *directrix*, and the fixed point, F, is called the *focus*. Hence, if $PF = PD$ for all positions of P, the point P describes a parabola. The point O, midway between the focus and the directrix, is called the *vertex* of the parabola.



A parabola has an axis of symmetry which passes through its vertex and its focus.

2. *Equation of a Parabola.* If the axis of a parabola is parallel to the y-axis, the parabola may be represented by an equation of the type

$$(1) \qquad y = ax^2 + bx + c.$$

If the axis is parallel to the x-axis, the equation of the parabola is of the type

$$(2) \qquad x = ay^2 + by + c.$$

When $b = c = 0$, the parabola passes through the origin, and its axis is the y-axis or the x-axis.

3. *Plotting a Parabola.* The graphs of numerous parabolas of type (1) were plotted in Chapter VII. It remains to consider parabolas whose equations are of type (2).

Illustrative Example. Plot the graph of $x = \frac{1}{4}y^2 - 1$.

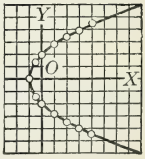
Solution. 1. The intercepts are ± 2 on the y-axis and -1 on the x-axis. (Why?)

2. Since the given equation may be written in the form $y^2 = 4x + 4$, we have $y = \pm 2\sqrt{x+1}$. (Explain.)

3. Using this equation, we obtain the following table:

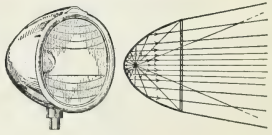
$x =$	0	-1	$-\frac{1}{2}$	1	2	3	4
$y =$	± 2	0	± 1.4	± 2.8	± 3.5	± 4	± 4.5

4. Plotting the points suggested by this table, and drawing a smooth curve through them, we obtain a parabola such as the one shown in the figure.



DISCUSSION. Values of x less than -1 would make y imaginary. (Explain.) Hence such values must be excluded. The curve is symmetric with respect to the x-axis. (Why?)

4. *Applications.* As in the case of the ellipse, there are many interesting applications of the parabola in the arts and sciences. It appears frequently in the arches of bridges. The cable of a suspension bridge, if loaded uniformly per horizontal foot, has the form of a parabola. Certain sections of headlights and, as a rule, the reflecting surfaces used in telescopes are parabolic. The path of a projectile, if air resistance is neglected, is a parabola. The graphs of many scientific formulas are parabolas.



The diagram at the right shows a cross-section of the automobile headlight at the left

The rays of light coming from a lamp placed about at the focus are reflected in lines very nearly parallel to the axis of the curve

Exercises. *The Parabola*

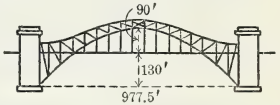
Plot the graph of each of the following equations:

1. $y^2 = x$.
2. $x = 3y^2$.
3. $x = -2y^2$.
4. $y^2 = x - 2$.
5. $x = y^2 - 4$.
6. $y^2 - 2x = 2$.
7. $x = y^2 + 3y + 2$.
8. $x = 3y^2 - 6y - 5$.
9. $y = 3x^2 - 6x - 5$.

10. Engineers make use of parabolas when building bridges. The one in the photograph has a parabolic steel arch with a span of 592 ft. and a rise of 100 ft. Using the vertex of the curve as the origin, write the equation of the parabola. If the roadway of the bridge is 47 ft. below the vertex of the curve and 44 ft. above high water, what is the horizontal distance between the points where the high-water mark touches the curve?

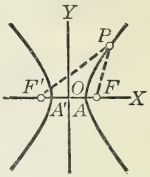


11. The accompanying diagram shows a bridge in the construction of which a parabolic steel arch was used. The span of the arch is 977.5 ft., and its height is 220 ft. Write the equation of this arch, using the vertex as the origin. How long is the portion of the roadway which is situated between the left and the right branch of this parabolic arch, if the roadway is about 130 ft. above the span of the arch?



132. The Hyperbola. 1. *Definition.* If a point P in a plane moves so that the *difference* between its distances from two fixed points, such as F and F' in the figure on page 274, remains constant, it describes a curve called a *hyperbola.*

This curve consists of two *branches*. The fixed points, F and F', are called the *foci*, and the points A and A' are called the *vertices* of the hyperbola. The point O, which bisects AA', is called the *centre* of the hyperbola. The line passing through A and A' is called the *principal axis* of the hyperbola.



A hyperbola such as the one shown in the figure is symmetric with respect to the x-axis and the y-axis.

2. *Equation of a Hyperbola.* If the centre of the hyperbola is at the origin, and if the hyperbola is symmetrically placed with reference to the x-axis and the y-axis, its equation has the form

$$ax^2 - by^2 = c.$$

The equation

$$xy = c$$

also represents a hyperbola, which, however, has different axes of symmetry.

3. *Plotting a Hyperbola.* The method used in plotting a hyperbola may be inferred from the following examples:

Illustrative Example 1. Plot the graph of $9x^2 - 4y^2 = 36$.

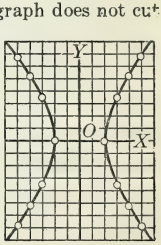
Solution. 1. If $x = 0$, y is imaginary. Hence the graph does not cut the y-axis. If $y = 0$, $x = \pm 2$.

2. The equation may be written in the form

$$y = \pm \tfrac{3}{2}\sqrt{x^2 - 4}.$$

3. Hence we obtain the following table:

$x =$	± 2	± 3	± 4	± 5
$y =$	0	± 3.4	± 5.2	± 6.9

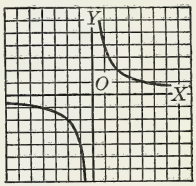


4. With the aid of this table we may draw a curve such as the one shown in the figure.

Discussion. Values of x lying between $+2$ and -2 would make y imaginary. (Why?) Hence these values must be excluded.

Illustrative Example 2. Plot the graph of $xy = 4$.

Solution. The given equation may be written in the form $y = \frac{4}{x}$. The value $x = 0$ must be excluded. (Why?) With the aid of a table based on this equation,* we may readily plot a hyperbola (see page 165) such as the one shown in the figure.



4. *Applications.* Hyperbolas may be used in computations which involve a constant product of two variable factors. The graphs of important scientific formulas, such as those arising in the study of gases and of electricity and magnetism, are hyperbolas. This curve also has a military importance. Hyperbolas assist in "sound ranging," that is, in locating an invisible source of sound such as a hidden gun.

Exercises. *The Hyperbola*

Plot the graph of each of the following equations:

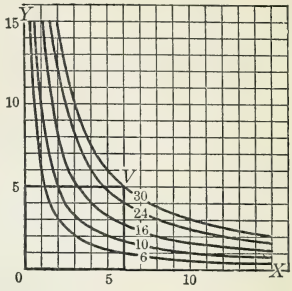
1. $x^2 - y^2 = 4$.
2. $x^2 - y^2 = 25$.
3. $y^2 - x^2 = 16$.
4. $xy = 5$.
5. $xy = -10$.
6. $y = \frac{6}{x}$.
7. $4x^2 - 9y^2 = 36$.
8. $y^2 = 9 + 4x^2$.
9. $3x^2 - 4y^2 = -12$.

EXPLANATION. Whenever the product of two variables, such as x and y, is constant, the relation between these variables can be pictured by means of a hyperbola. This fact can be used to make certain computations by means of a graphic chart like the one shown on page 276. The successive hyperbolas (one branch only being drawn in each case) represent the graphs of $xy = 6$, $xy = 10$, $xy = 16$, and so on.

* As the values of x become smaller, approaching the value 0, the values of y increase. Similarly, as the values of y become smaller, the values of x increase. Hence the branches of the hyperbola *approach* the y-axis and the x-axis more and more closely, without ever reaching them.

Suppose, now, that a printer or an engraver wishes to determine quickly the price of a rectangular half-tone or zinc etching of specified size. He places a card representing the half-tone so that two of its sides fall along the x- and y-axes. Then, if the fourth vertex (V) falls on any one of the hyperbolas, such as the one representing $xy = 30$, it is evident that the area of the card is 30 square units.

The price corresponding to each area is indicated on the hyperbola concerned, so that the printer can at once quote the price for a plate of any given size falling within the range of the chart.



10. Make a graphic chart such as the one shown above, using a larger scale. With the aid of this chart determine the areas of several rectangular cards. Verify each result by measuring the length and the width of the card and then obtaining the area by multiplication.

11. Boyle's Law states that the product of the *volume* and the *pressure* of a gas at a given temperature is constant. That is, in symbols, $pv = c$. If $p = 90$ lb. and $v = 20$ cu. ft., draw a graph showing other values of p and v that would result in the same value of c. (Use any convenient scale.)

133. General Equation of the Second Degree in Two Variables. The general form of an equation of the second degree in x and y is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

By assigning special values to the coefficients, including the value zero, we may readily obtain equations of the types considered in the preceding pages.

In general, *a second-degree equation in two variables represents a conic section,* provided the term "section" is understood to include certain "special cases" that may arise.

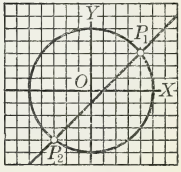
For example, the equation $x^2 - y^2 = 0$ is of the type which represents a hyperbola. Observe, however, that the equation $x^2 - y^2 = 0$ is equivalent to $(x + y)(x - y) = 0$ and hence corresponds to the pair of equations $x + y = 0$ and $x - y = 0$. Hence the graph of $x^2 - y^2 = 0$ is represented by the two straight lines which correspond to the equations $y = x$ and $y = -x$. Geometrically, such a section arises when a plane passes through the vertex of a double cone and contains an element of the conical surface.

In like manner, the equation $x^2 - 5xy + 6y^2 = 0$ may be written in the form $(x - 3y)(x - 2y) = 0$. Hence its graph consists of the pair of lines corresponding to $x - 3y = 0$ and $x - 2y = 0$.

2. Graphic Solution of Pairs of Equations involving Quadratics

134. One Quadratic and One Linear Equation. The figure shows the graphs of the pair of equations

$$(1)\quad x^2 + y^2 = 25,$$
$$(2)\quad x - y = 1.$$

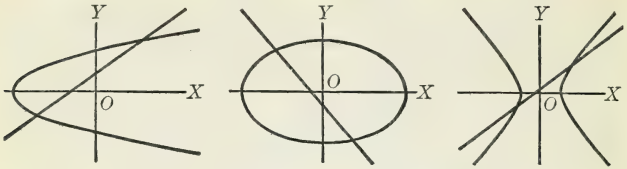


These graphs intersect at the points P_1 and P_2. The coördinates of P_1 are $x = 4$ and $y = 3$, and those of P_2 are $x = -3$ and $y = -4$. It is evident that both pairs of coördinates satisfy each of the given equations. Hence, corresponding to the two points of intersection, we have *two sets of roots,* which may be tabulated as shown in the table at the right.

Roots:	P_1	P_2
$x =$	4	-3
$y =$	3	-4

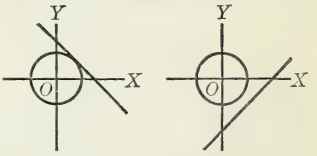
Moreover, since the two graphs have no other point in common, these two sets of values represent the *complete solution* of the given pair of simultaneous equations.

Instead of the equation of a circle, we might have occasion to use the equation of one of the other conic sections, as suggested in the diagrams below.



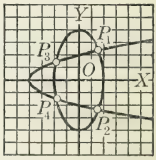
Now, we shall regard it as self-evident that *any one of these four curves can be cut by a straight line in not more than two points.* Hence it follows that *if one of a pair of equations is quadratic and the other is linear, there cannot be more than two distinct real solutions.*

NOTE. The adjoining figures suggest "special cases" that may arise. Thus, the straight line may be *tangent* to the curve, or it may not touch the curve at all. In the first case there is but one *real* solution, since the two pairs of real values have become identical. In the second case we shall find that the required roots are *imaginary.*



135. Two Quadratic Equations. Let it be required to solve graphically the pair of equations

$$(1)\quad 4x^2 + y^2 = 16,$$
$$(2)\quad y^2 = x + 4.$$

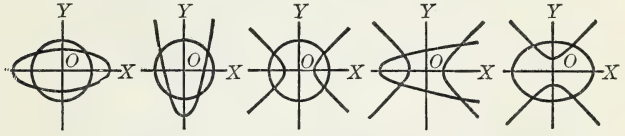


The graphs of these equations, an ellipse and a parabola, intersect in *four* points. The coördinates of each point represent a solution of the given pair of equations. Hence we find that the *complete solution* is represented by the four sets

of coördinates of the points of intersection. These resulting four pairs of corresponding values of x and y may be tabulated as follows:

Roots:	P_1	P_2	P_3	P_4
$x =$	1.6	1.6	− 1.9	− 1.9
$y =$	2.4	− 2.4	1.5	− 1.5

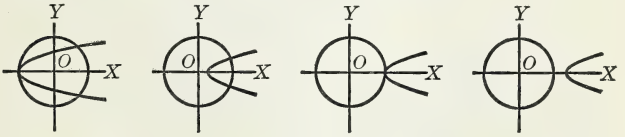
Since a given pair of equations might involve any two of the conic sections, we may have graphic combinations such as the following:



(1) Circle, Ellipse | (2) Circle, Parabola | (3) Circle, Hyperbola | (4) Parabola, Hyperbola | (5) Ellipse, Hyperbola

Any two of these curves can intersect in not more than four points. Hence it follows that *a system of two quadratic equations in two variables cannot have more than four distinct real solutions.*

NOTE. Special cases may, however, arise, as the following figures suggest:



That is, there may be *three* distinct *real* solutions, *two* distinct *real* solutions, *one real* solution, or *no real* solution. Two or four of the solutions may be imaginary, and two real solutions may be identical.

Exercises. *Graphic Solution of Pairs of Equations involving Quadratics*

Solve graphically each of the following pairs of equations. If the graphs do not intersect, designate the roots as imaginary.

1. $x^2 + y^2 = 25$,
   $x + y = 1$.
2. $y = x^2 + 3$,
   $7x + 42 = 8y$.
3. $x^2 + y^2 = 25$,
   $3x + 4y = 25$.
4. $y = x^2 + 3$,
   $y = 4x - 1$.
5. $y = x^2 + 3$,
   $y - x = 1$.
6. $x^2 + y^2 = 9$,
   $x - y = 7$.
7. $4x = 9y^2 - 16$,
   $3x - y = 3$.
8. $x^2 + 4y^2 = 16$,
   $y = 2x + 1$.
9. $x^2 - y^2 = 4$,
   $3y - x = 6$.
10. $xy = 5$,
    $y = x + 3$.
11. $xy = -4$,
    $y = 2x + 1$.
12. $x^2 + y^2 = 9$,
    $4x^2 + 16y^2 = 64$.
13. $x^2 + y^2 = 3$,
    $x^2 + 5y^2 = 20$.
14. $x^2 + y^2 = 4$
    $2x^2 = y + 3$.
15. $x^2 + y^2 = 5$,
    $y^2 = 4x$.
16. $x^2 + y^2 = 8$,
    $xy = 2$.
17. $x^2 + 3y^2 = 12$,
    $y = 2x^2 - 3$.
18. $x^2 + y^2 = 6$,
    $x^2 - y^2 = 9$.
19. $2x^2 + y^2 = 16$,
    $x = 2y^2 - 5$.
20. $4x^2 + 9y^2 = 36$,
    $x^2 - y^2 = 4$.
21. $y = 4x^2 - 5$,
    $x = 3y^2 - 4$.
22. $x^2 + 4y^2 = 4$,
    $4x^2 + y^2 = 4$.
23. $x^2 - y^2 = -9$,
    $xy = 1$.
24. $x^2 - y^2 = 1$,
    $y = -2x^2 + 4$.

3. Algebraic Solution of Pairs of Equations involving Quadratics

136. Systems involving One Quadratic and One Linear Equation. The graphic method of solving a system* of equations involving quadratics serves to indicate clearly the nature of the required roots. As in the case of linear equations, however, it is not always convenient, especially when the roots have decimal values. An algebraic solution, as a rule, furnishes the numerical values of the roots more readily. It will be seen that the algebraic methods used in the following pages are very similar to those employed in Chapter II.

Illustrative Example. Solve the pair of equations

$$(1)\quad x - 2y = 2,$$
$$(2)\quad x^2 - y^2 = 4.$$

Solution. 1. In equation (1), expressing x in terms of y, we have

$$(3)\quad x = 2y + 2.$$

2. Substituting for x in equation (2), we have

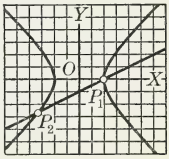
$$(2y + 2)^2 - y^2 = 4.$$

3. Then $4y^2 + 8y + 4 - y^2 = 4,$

or $3y^2 + 8y = 0.$

4. That is, $y(3y + 8) = 0,$

and $y = 0 \text{ or } -\frac{8}{3}.$



5. Substituting 0 for y in equation (1), we have $x = 2$. Also, substituting $-\frac{8}{3}$ for y in equation (1), we have $x = -\frac{10}{3}$.

6. The two sets of roots thus obtained may be arranged as shown in the adjoining table.

Roots:	(1)	(2)
$x =$	2	$-\frac{10}{3}$
$y =$	0	$-\frac{8}{3}$

Check. Substitute these two pairs of values for x and y in each of the given equations. It will readily be seen that the two sets of roots obtained above agree with the coördinates of P_1 and P_2, the points of intersection of the graphs picturing the given equations.

*A set of simultaneous equations in n unknowns is usually referred to as a "system" of equations in n variables.

Exercises. *One Quadratic and One Linear Equation*

Solve each of the following pairs of equations, indicating each set of roots and checking each solution:

1. $x^2 + y^2 = 106$, $x = 14 - y$.
2. $x^2 + y^2 = 73$, $x + y = 11$.
3. $x - y = 6$, $xy = 16$.
4. $2x + 3y = 23$, $xy = 20$.
5. $x^2 - y^2 = 5$, $3x - y = 7$.
6. $x^2 - y^2 = 21$, $x = \frac{5}{2}y$.
7. $x^2 + 2y^2 = 6$, $2x = 3 + y$.
8. $3x^2 + 4y^2 = 64$, $x - 2y = 0$.
9. $y = x^2 + 5$, $y - x = 7$.
10. $3x - 5y = -21$, $y - 2x^2 = 4$.
11. $x + 5 = 4y$, $x^2 + y^2 = 25$.
12. $5y = 3x + 16$, $xy = 15$.
13. $x^2 - 5 = 4y^2$, $y + 5 = 2x$.
14. $4x^2 + 25y^2 = 2$, $5y = 2 - 2x$.
15. $y^2 - 11x = 9$, $y + 2x = 2$.
16. $x^2 + xy = 15$, $3x - 2y = 5$.
17. $x^2 + y^2 + 2y = 9$, $x - y = 5$.
18. $x^2 - xy + y^2 = 21$, $y = 9 - x$.
19. $x^2 + xy + y^2 = 49$, $y - x + 2 = 0$.
20. $x^2 - 4xy - 5 = 0$, $2x - 3y - 7 = 0$.
21. $y + 10x - 3xy = 0$, $y = x + 2$.
22. $5x^2 - 2 = xy$, $2x + y - 5 = 0$.
23. $x = 7 - 2y$, $\frac{3}{x} + \frac{10}{y} = 1$.
24. $x - 2y + \frac{2y}{x} = 1$, $x + y = 4$.
25. $x^2 + y^2 = m$, $x + y = n$.
26. $xy = a$, $x + y = b$.
27. $2x + 3y = 24$, $\frac{x^2}{9} + \frac{y^2}{8} = 6$.

137. Systems involving Two Quadratic Equations. The algebraic solution of such pairs of equations can usually be effected by methods of elimination similar to those discussed previously, or by using special devices that can best be inferred from the suggestions and illustrations given below. In all cases it is necessary to indicate clearly the pairs of roots which represent common solutions of the given system of equations.

Illustrative **Example 1.** Solve the pair of equations

$$(1)\quad x^2 + y^2 = 20,$$
$$(2)\quad xy = 8.$$

Solution. 1. In equation (2), expressing x in terms of y, we have

$$(3)\qquad x = \frac{8}{y}.$$

2. Substituting for x in equation (1), we obtain

$$\frac{64}{y^2} + y^2 = 20.$$

3. Then $$64 + y^4 = 20\,y^2,$$

and $$y^4 - 20\,y^2 + 64 = 0.$$

4. Factoring completely, we have

$$(y + 2)(y - 2)(y + 4)(y - 4) = 0.$$

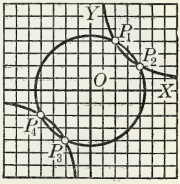
5. Hence $$y = \pm 2 \text{ or } \pm 4.$$

6. Substituting each of these values for y in equation (2), we have

$$x = \pm 4 \text{ or } \pm 2.$$

7. The four solutions thus obtained may be tabulated as follows:

Roots:	(1)	(2)	(3)	(4)
$x =$	2	4	-2	-4
$y =$	4	2	-4	-2



Check. Substitute these four pairs of values for x and y in each of the given equations. It will readily be seen that the four sets of roots agree with the coördinates of P_1, P_2, P_3, and P_4, the points of intersection of the graphs picturing the given equations.

Illustrative Example 2. Solve the pair of equations

$$(1)\ 2x^2 + 3y^2 = 21,$$
$$(2)\ 3x^2 - 4y^2 = 23.$$

Solution. 1. Multiplying both members of equation (1) by 3, and both members of equation (2) by 2, we have

$$(3)\ 6x^2 + 9y^2 = 63,$$

and

$$(4)\ 6x^2 - 8y^2 = 46.$$

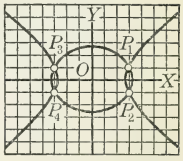
2. Subtracting equation (4) from equation (3), and solving the resulting equation, we find that $y = \pm 1$.

3. Substituting $+1$ for y in equation (2), we find that $x = \pm 3$.

4. Substituting -1 for y in equation (2), we find that $x = \pm 3$.

5. Writing the four solutions thus obtained in tabular form, we have

Roots:	(1)	(2)	(3)	(4)
$x =$	3	3	-3	-3
$y =$	1	-1	1	-1



Check. Substitute these four pairs of values for x and y in each of the given equations. Do these solutions agree with those suggested by the graphs of the given equations?

Exercises. *Systems involving Two Quadratic Equations*

Solve each of the following pairs of equations, indicating each set of roots, and checking each set of real roots:

1. $x^2 + y^2 = 65$,
$xy = 28$.

2. $x^2 + y^2 = 89$,
$xy = 40$.

3. $x^2 + y^2 = 29$,
$xy = -10$.

4. $x^2 - xy + y^2 = 13$,
$xy - 4 = 0$.

5. $x^2 - y^2 = 24,$
$xy = 35.$

6. $x^2 + y^2 = 29,$
$2x^2 + y^2 = 33.$

7. $x^2 + y^2 = 50,$
$x^2 - y^2 = -48.$

8. $2x^2 - y^2 = 2,$
$x^2 + 2y^2 = 41.$

9. $6x^2 - y^2 - 5 = 0,$
$x^2 + 2y^2 - 107 = 0.$

10. $4x^2 + 3y^2 = 112,$
$3x^2 - 2y^2 = -69.$

11. $x^2 + y^2 = 40,$
$x + 2y^2 = 74.$

12. $y^2 - 2x^2 = 7,$
$4y + 3x^2 = 47.$

13. $11x^2 + 5y^2 = 301,$
$4x^2 - 3y^2 = -11.$

14. $4x^2 = 5xy + 46,$
$3x^2 = 2xy + 24.$

15. $y^2 = 8 - x,$
$y^2 = 16 - 17x.$

16. $2x^2 - 3y^2 = 29,$
$x^2 - y = 15.$

17. $x^2 + 3y^2 = 27 - xy,$
$2x^2 - 3y^2 = xy.$

18. $x^2 + y^2 = 16,$
$x^2 - 6y = 0.$

19. $3x^2 + 4y = 34,$
$7x^2 - 2y = 0.$

20. $5xy + y^2 = 43 - x^2,$
$5xy - y^2 = 25 - x^2.$

21. $4x^2 - 25y^2 = 0,$
$2x^2 - 10y^2 = 3y + 4.$

22. $x^2 = 3 - y^2 - xy,$
$2x^2 = 9 + y^2 + xy.$

23. $x^3 + y^3 = 9,$
$x + y = 3.$

SUGGESTION. Factor the left member of the first equation.

24. $x^3 + y^3 = 126,$
$x^2 - xy + y^2 = 21.$

SUGGESTION. Factor the left member of the first equation.

25. $x^3 - y^3 = 19,$
$x - y = 1.$

26. $x^3 - y^3 = 56,$
$x^2 + xy + y^2 = 28.$

27. $x^2 - xy + y^2 = 13,$
$x^3 + y^3 = 91.$

28. $y = 7 - x,$
$y^3 = 133 - x^3.$

Exercises. *Problems*

1. Two numbers are in the ratio of 3 : 2, and the difference of their squares is 45. Find the numbers.

2. The total area of two square pieces of tin is 58 sq. ft. A side of the larger square is 4 ft. longer than a side of the smaller square. Find the length of the side of each square.

3. The altitude to the base of an isosceles triangle is 12 in., and the perimeter of the triangle is 36 in. Find the length of each side of the triangle.

4. What are the dimensions of a rectangular lot that contains 10 A. and requires 164 rd. of fencing to inclose it?

5. The smaller of two cubical boxes is placed on top of the larger. Their combined height is 5 ft., and their combined volume is 35 cu. ft. Find the length of an edge of each cube.

6. If the numerator of a fraction is increased by 2 and the denominator is increased by 3, the value of the resulting fraction is $\frac{3}{4}$. The sum of the original fraction and its reciprocal is $2\frac{1}{20}$. Find the original fraction.

7. Find a number consisting of two digits such that the tens' digit exceeds the units' digit by 3 and the number is 9 more than the sum of the squares of its digits.

8. A merchant bought a number of mirrors for \$100. Four were broken. He sold the remainder at \$3 apiece more than they cost him, and thus received \$47 more than he had paid for the mirrors. How many mirrors did he buy?

9. A dealer found that if the price of lemons were lowered 6¢ a dozen, ten more lemons could be purchased for \$1. What was the original price per dozen?

10. A motorist drove a distance of 210 mi. On the return trip he averaged 6 mi. an hour more and made the trip in 1 hr. 10 min. less time. What was his average rate when going and when returning?

11. Two motorists meet at the intersection of two roads which are perpendicular to each other. If they travel in opposite directions from this point, they will be 35 mi. apart at the end of 30 min. But if they travel so that their routes are at right angles to each other, they will be 25 mi. apart at the end of 30 min. Find the rate of each.

12. Find two numbers such that their sum, their product, and the difference of their squares shall all be equal.

13. It cost 80¢ a square yard to pave a rectangular court and \$1 a yard to build a fence around it. The total cost was \$100. If the length of the court is 2 yd. more than the width, what are the dimensions of the court?

14. A rectangular swimming pool is surrounded by a walk 6 ft. wide. The area of the walk is 2256 sq. ft. and the combined area of the pool and the walk is 9600 sq. ft. Find the dimensions of the swimming pool.

15. A rectangular court can be paved with 200 square tiles of a certain size. But if a square tile with a side 1 in. longer is used, only 128 tiles are needed. Find the length of a side of each kind of tile.

16. Find the length of the altitude upon the longest side of a triangle whose sides are 11 in., 13 in., and 20 in.

17. A motor boat takes $1\frac{1}{2}$ hr. longer when going upstream a distance of 24 mi. than it does when returning downstream. If the rate of the boat in still water were 2 mi. an hour faster, the rate downstream would be 18 mi. an hour. Find the rate of the boat in still water and the rate of the stream.

18. At simple interest, \$1500 amounted to \$1875 in a certain time and at a certain per cent. If the time had been 1 yr. longer and the rate 1% higher, the amount would have been \$2040. Find how many years the money was at interest.

138. Summary. In this chapter you have learned

(1) How to plot the graph of an equation representing one of the conic sections.

(2) How to solve graphically pairs of equations involving quadratics.

(3) How to solve algebraically systems involving one linear and one quadratic equation.

(4) How to solve algebraically systems involving two quadratic equations.

(5) How to solve problems by using systems of equations involving quadratics

Test on Chapter IX

In the case of each of the following equations state what kind of curve results when the equation is represented graphically:

1. $x^2 + y^2 = 10$.
2. $3xy = 5$.
3. $2x^2 + y^2 = 18$.
4. $y^2 = x + 5$.
5. $y = x^2 - 3x - 18$.
6. $4x^2 - y^2 = 16$.

Solve algebraically each of the following pairs of equations. Check each solution.

7. $x^2 + 2y^2 = 9$,
   $x = 5 - 2y$.
8. $2x - y = 0$,
   $y - x^2 = 3x - 2$.
9. $xy + x^2 = 24$,
   $y = 3x - 4$.
10. $x^2 + y^2 = 25$,
    $2x + y^2 = 10$.
11. $2x^2 + 3y^2 = 30$,
    $3x^2 - 2y^2 = 19$.
12. $x^2 + y^2 = 5$,
    $xy = 2$.

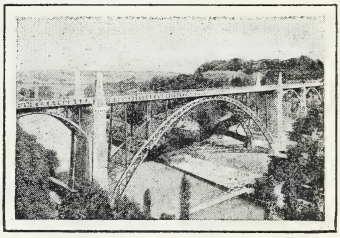
13. Write the equation of a circle whose centre is the origin and whose radius is $\sqrt{5}$.

14. Give the equation of a parabola symmetric with respect to the x-axis and intersecting the x-axis at $x = -2$.

15. Write the equation of an ellipse symmetric with respect to the x-axis and the y-axis and having the intercepts $x = \pm 3$ and $y = \pm 2$.

16. The area of a rectangle is 48 sq. in., and its diagonal is 10 in. What are the dimensions of the rectangle? (Solve this problem graphically.)

17. A bridge over the river Aar, at Bern, Switzerland, has a parabolic arch with a span of 384 ft. and a height of 104 ft. Using the vertex of the curve as the origin, write the equation of the parabola. Assuming that the roadbed is 15 ft. above the vertex of the parabola and that the vertical supports are 32 ft. apart, determine the lengths of the four longest supports from the arch to the roadbed.



Exercises. *Review*

1. Given the formula $r = \sqrt[3]{\frac{3\,V}{4\,\pi}}$. Using logarithms, find the value of r when $V = 904.32$ and $\pi = 3.14$.

2. Find the value of $\log 6435 - \log 0.06435$.

3. If $\log a = 0.5378$ and $\log b = 9.7853 - 10$, find $\log a^3b^2$.

4. Given $\log x^3 = 0.7659$; find $\log 100\,x$.

5. Given $\log 3 = 0.4771$ and $\log 4 = 0.6021$; find, without using tables, the value of $\log \sqrt[5]{12}$.

6. Complete this statement: $\log 243 = (\quad) \log 3$.

Find the value of each of the following:

7. $64^{\frac{2}{3}} + (8\,n)^0 - 0.00032^{\frac{2}{5}}$.

8. $27^{-\frac{1}{3}} - 4^{\frac{3}{2}} - 3\,n^0 + \dfrac{1}{16^{-\frac{3}{4}}}$.

9. $2\,x^{-\frac{1}{3}} + \frac{1}{2}\,x^{\frac{2}{3}} + 4\,x^0$ when $x = 8$.

10. The sum of the roots of an equation is 7 and their difference is 3. Write the equation.

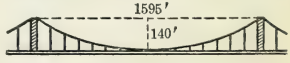
11. If $2\,x^2 - 7\,x + 3 = 0$, by what amount does the sum of the roots exceed the product of the roots?

12. What extraneous root is introduced into the equation $4\,x - 1 = x + 2$ by multiplying each member by $x - 3$?

13. The divisor is $a - 2\,c$, the quotient is $a - 3\,b$, and the remainder is $-6\,bc$. What is the dividend?

14. Use the formulas $s = \frac{1}{2}\,gt^2$ and $v = gt$ to obtain a formula for s that does not contain g.

15. The cables of the Brooklyn Bridge support a roadway which is nearly 1595 ft. long between the suspension piers. The roadway is attached to the cables by vertical rods. The piers extend to a height of 140 ft. above the roadway. Write the equation of the parabola formed by the cables. (Assume that the vertex of the curve is at the origin and level with the roadway.)



16. In Exercise 15, find the lengths of the first three of the supporting rods, which are at intervals of 100 ft., measured horizontally from one of the piers.

17. The difference in the cubes of two consecutive integers is 469. Find the numbers.

18. The area of a right-angled triangle is 720 sq. ft. and its perimeter is 180 ft. Find the length of the hypotenuse.

19. An automobile radiator whose capacity is $4\frac{1}{2}$ gal. is filled with a mixture of water and alcohol which is 10% alcohol. How much of the mixture must be drawn off and replaced by alcohol in order that the radiator may contain a mixture that is 30% alcohol?

20. The inside dimensions of a picture frame are 18 in. and 20 in. The area of the frame is 168 sq. in. Find the width of the frame.

Mathematics is the most marvellous instrument created by the genius of man for the discovery of truth.—LAISANT

Mathematics is the indispensable instrument of all physical research.—BERTHELOT

It is in mathematics we ought to learn the general method always followed by the human mind in its positive researches.
COMTE

GENERAL SUMMARY AND REVIEW

1. Planning a Final Review. In connection with your final review, at the completion of your Second Course in algebra, it will help you to read again the summary and the test at the end of each chapter. You may then study the General Summary which appears below. The review exercises which follow this outline of the entire course will enable you to find out to what extent you understand and can apply the fundamental principles, the processes, and the ideas of algebra.

GENERAL SUMMARY

I. *The Language and the Fundamental Laws of Algebra*

1. The meaning and use of algebraic symbols and terms.
2. The types of number included in algebra.
3. The laws of order, of grouping, of signs, and of distribution.

II. *The Technique of Algebra*

1. The four fundamental operations involving integral algebraic expressions.
2. Special products and factoring. The Factor Theorem.
3. Algebraic fractions:
   (1) The four fundamental operations.
   (2) Complex fractions.
4. Powers and roots:
   (1) The fundamental laws of exponents and their use.
   (2) Meaning and use of zero, negative, and fractional exponents.

(3) The square root of numbers.
(4) The fundamental laws of radicals and their use.
(5) Meaning of imaginary numbers.

5. Logarithms:
   (1) The fundamental laws.
   (2) Computation by means of logarithms.
   (3) The use of logarithms in the solution of exponential equations.
   (4) The use of antilogarithms.

III. *The Equation*

1. Equations of the first degree in one unknown.
2. Fractional equations. Ratio and proportion. Variation.
3. Sets of equations of the first degree.
4. Quadratic equations in one unknown:
   (1) Solution (*a*) by factoring, (*b*) by completing the square, (*c*) by formula.
   (2) Introduction of extraneous roots.
   (3) Relation between roots and coefficients.
   (4) Formation of an equation from its roots.
   (5) Character of the roots.
5. Pairs of equations involving quadratics.
6. Simple exponential equations.
7. Simple radical equations.
8. Solution of verbal problems by means of equations.

IV. *The Formula*

1. Meaning and use of formulas arising in mathematics, in science, and in everyday life.
2. Construction from specified conditions or from tables.
3. Transformation of formulas.
4. Graphic representation of algebraic formulas (or functions) of the first degree. of the second degree, or of a higher degree (cubic functions, exponential and logarithmic functions).

V. *The Graph*

1. Meaning and importance of graphic representation.
2. Graphic treatment of the linear function.
3. Graphic treatment of the quadratic function.
4. Graphic treatment of higher functions.
5. Graphic solution of pairs of linear equations.
6. Graphic solution of pairs of equations involving quadratics.
7. Graphic solution of verbal problems.

VI. *Problems*

2. General Review Exercises. The following groups of exercises stress the types of work which have received the major emphasis throughout the course. Many of these exercises were suggested by questions in recent examinations prepared by leading examining bodies.

I. Algebraic Technique

1. Fundamental Operations

Find the value of each of the following:

1. $8 \times 3^2 \div 6 + 4^3 - 12 \times \frac{1}{2} + 7$.

2. $(12 - 4 \times 2)(5 - 7) + 24 \div 6 \times 5 - (-8)$.

If $A = 3a^2 - 3ab + 4b^2$, $B = 2b^2 + a^2$, $C = 18ab - 5a^2 + b^2 + 8$, *and* $D = 6a^4 - 12b^4 - 7a^2b^2 + 17ab^3$, *find:*

3. $A + B - C$.

4. $B - (A + C)$.

5. $B \times C$.

6. $D \div B$.

7. $A \times B + D$.

8. $D \div A - C$.

Remove parentheses and combine similar terms in each case:

9. $9a^2 - 3(-a + 2b^2) + 5(a - b)^2$.

10. $9x^3 - 5x[3x - (-2x^2 + 6) - 12x]$.

11. $4x - \{3.6y - 0.2[1.4x + 5y - (0.3y + 1.5x)]\}$.

In Exercises 12 and 13, inclose the last three terms in each case in parentheses preceded by a minus sign:

12. $4\,c^2 - 3\,x^2 - 12\,x^2y^2 + 7\,y^2$.

13. $81\,x^4 - 25\,r^4 - 30\,r^2s^2 - 9\,s^4$.

14. If $m = 2$ and $n = 3$, find the value of $(2\,m)^3n^2$ and of $(-3\,m^2)(-n^3)$.

Factor completely each of the following:

15. $0.04\,x^2 - 0.0025$.

16. $a^3 - 3\,a + 2$.

17. $9\,r^{6n} - 30\,r^{3n} + 25$.

18. $25\,c^{2x} - 2\,c^x + 0.04$.

19. $6\,a^3 - 24\,a^9n^2$.

20. $5\,y^2 - 5\,y + 3\,a(1 - y)$.

21. $64\,x^6 - y^6$.

22. $a^{8n} - 16\,b^{8n}$.

23. $d^3 + d^2 - d - 1$.

24. $n^2 + 36 - 12\,n - 49\,s^2$.

25. $n^4 - 25\,n^2 - 60\,n - 36$.

26. $a^3 - 2\,a^2 - a - 6$.

27. $2\,y^{3n} + 0.016$.

28. $x^{n+3} - 7\,x^{n+1} - 6\,x^n$.

29. $36\,c^2 - (7\,d - 1)^2$.

30. $27\,x^3 - 1000^{-1}$.

31. $a(a + 1)(4\,a - 5) - 6(a + 1)$.

32. $4(m - n)^2 - 12\,r(m - n) + 9\,r^2$.

33. If $2\,c - 3$ is a factor of $kc^2 - 7\,c - 3$, what is the value of k?

In each of the following cases perform the indicated operations and simplify the result:

34. $\dfrac{x^2}{x+y} - \dfrac{y^2}{x-y} - (x - y)$. Check, letting $x = 3$ and $y = 2$.

35. $\dfrac{2\,n^2 - 18}{n^2 + 4\,n - 21} - \dfrac{n - \frac{1}{2}}{n - 3}\left(3 - \dfrac{n + 27}{n + 7}\right)$. Check, letting $n = 5$.

36. $\left(\dfrac{d}{c-d} + 1\right) \div \left(\dfrac{c}{d-c}\right)$.

37. $\left(\dfrac{1}{c^2} - \dfrac{1}{d^2}\right) \div \left(\dfrac{c}{d} - \dfrac{d}{c}\right)$.

38. $\dfrac{2(bx + ax^2)}{1 - 4\,x^2} + \dfrac{3\,ax^2 + 3\,b}{6\,x - 3} - \dfrac{ax^2 - b}{2\,x + 1}$.

39. $\left(c - \dfrac{2\,d^2 - c^2 + cd}{2\,d}\right) \cdot \dfrac{6\,c^2 + 5\,cd - d^2}{6\,c^2 + 11\,cd - 2\,d^2} \div \left(\dfrac{c^3}{d} - cd\right)$.

40. $1 + \dfrac{2x}{2x+1} - \dfrac{8x^2-2}{1-4x^2}$.

41. $\left[(x^2-y^2) + \dfrac{4xy(y+x)}{x-y}\right] \div \dfrac{x^2+y(y+2x)}{4x^2-6xy+2y^2}$.

42. $\dfrac{\dfrac{a^3}{b^3}+1}{1+\dfrac{a}{b}}$.

43. $\dfrac{x-\dfrac{x^2-1}{x-1}}{3-x}$.

44. $\dfrac{\left(x+\dfrac{1}{y}\right)\left(y-\dfrac{1}{x}\right)}{\dfrac{x^2y^2-1}{3xy}}$.

45. $1 + \dfrac{c}{1+c+\dfrac{2c^2}{1-c}}$.

46. Evaluate $\dfrac{x+1}{2x^2-x-1} + \dfrac{x-1}{2x^2-3x-2}$ when $x = \frac{1}{3}$.

47. Substitute $\dfrac{b+c}{2}$ for x in $\left(\dfrac{x-b}{x-c}\right)^3 - \dfrac{x-2b+c}{x+b-2c}$.

2. Exponents, Radicals, and Logarithms

Find the numerical value of each of the following:

1. 100^0.
2. $(0.5)^0$.
3. 6^{-2}.
4. $9^{\frac{5}{2}}$.
5. $(0.125)^{\frac{1}{3}}$.
6. $81^{-\frac{3}{4}}$.
7. $(0.25)^{-\frac{3}{2}}$.
8. $(-\frac{8}{27})^{-\frac{1}{3}}$.
9. $3^{-3} \times 4^{-3}$.
10. $7^{\frac{3}{4}} \times 7^{\frac{1}{4}}$.
11. 4.16×10^{-5}.
12. $3^0 - 3^{-2}$.
13. $\sqrt{\dfrac{81^a}{9^{2a}}}$.
14. $\sqrt[r]{\dfrac{16}{2^{4-r}}}$.
15. $\sqrt{6} \times \sqrt{24}$.
16. $\sqrt{6} \div \sqrt{24}$.
17. $\dfrac{3^{-1} \times 3^0}{3^{-1}+3^{-2}}$.
18. $\left(\dfrac{3^2-6^0}{9^{-\frac{3}{2}}}\right)^{-\frac{2}{3}}$.
19. $\sqrt[8]{\dfrac{3^7+3^7+3^7}{4^7+4^7+4^7+4^7}}$.

Write each of the following in its simplest form with positive exponents and rational denominators:

20. $11(x^0+y^0)(2\sqrt{5}+3)^{-1}$.
21. $3x^0 - (3x)^0 - 1^{3x}$.
22. $3\sqrt{8} - 4\sqrt{\frac{1}{2}} + 2\sqrt{\frac{1}{2}}$.
23. $(-6a^3b^{\frac{3}{2}})\left(\dfrac{a^{-1}b^{-\frac{1}{2}}}{-3}\right)$.
24. $\dfrac{8^{\frac{2}{3}}a^{-4}b^0 - 9b^{\frac{4}{3}}}{2a^{-2} - 9^{\frac{1}{2}}a^0b^{\frac{2}{3}}}$.

25. $(\frac{1}{16})^{-\frac{1}{2}}(8^{\frac{3}{3}} - 3x^0)$.

26. $(5x^{\frac{1}{3}}y^4z^{a+2})(-3x^{\frac{2}{3}}y^3z^{a-2})$.

27. $(\sqrt{2}-\sqrt{3})^2+(\sqrt{2}+\sqrt{3})^2$.

28. $(x+1)\sqrt[3]{16x^2}+4x\sqrt[3]{\frac{1}{4}x^2}$.

29. $(a^{-\frac{3}{2}}+b^{\frac{3}{2}})^2-2\left(\sqrt{\frac{a}{b}}\right)^{-3}$.

30. $\dfrac{(y-5)^{\frac{1}{2}}+(y-5)^{-\frac{1}{2}}}{y-4}$.

31. $\dfrac{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y}{\sqrt{x}-\sqrt{y}}$.

32. $(27a^0+108^{\frac{1}{2}})3^{-2}+(\sqrt{3}-1)(2\sqrt{3}+3)^{-1}$.

33. Find the value of $4x^{-\frac{2}{3}}+3x^{\frac{1}{3}}+2x^0$, when $x=8$.

34. If $2x^{\frac{1}{2}}=1$, find the value of x^{-1}.

35. Rationalize the denominator of each of the following fractions:

(1) $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ (3) $\dfrac{7\sqrt{2}+\sqrt{3}}{7\sqrt{2}-\sqrt{3}}$

(2) $\dfrac{3\sqrt{3}+2\sqrt{5}}{3\sqrt{3}-2\sqrt{5}}$ (4) $\dfrac{a+b-c}{\sqrt{a+b}+\sqrt{c}}$

36. Evaluate $\left(\dfrac{ab^{-1}}{a^{-\frac{3}{4}}b^{\frac{1}{2}}}\right)^{-2}\div\left(\dfrac{b^{-2}}{a^{-1}}\right)^{-\frac{1}{2}}$ when $a=3$ and $b=\sqrt{3}$.

37. Simplify $10\sqrt{\frac{2}{5}}+9\sqrt{\frac{5}{3}}-\sqrt{60}-\dfrac{30}{\sqrt{15}}$ and find the value of the result to the nearest hundredth.

38. Find the product of $2+\sqrt{5}$ and its conjugate.

39. The area of a square is 5 sq. ft. Find the perimeter and the length of the diagonal.

40. The volume of a cube is 400 cu. cm. Find its total surface.

41. Which is greater, $5\sqrt{2}$ or $2\sqrt{5}$, and by how much? $10\sqrt{0.2}$ or $\sqrt{2}$? $\sqrt{5}$ or $10\sqrt{0.05}$?

42. Using the decimal value of each radical, arrange in the order of magnitude: $4\sqrt{2}$, $2\sqrt{3}$, $3\sqrt[3]{4}$, $\sqrt[3]{10}$.

43. The formula for the area of a regular octagon of side s is $A=2s^2(\sqrt{2}+1)$. Express this formula in decimal form. Evaluate A if $s=10$ in.

Given $2 = 10^{0.30103}$, $4.81 = 10^{0.6821}$, *and* $5.96 = 10^{0.7752}$, *express each of the following as a power of 10:*

44. 20.
45. 2×10^5.
46. 4.
47. 64.
48. $\sqrt{2}$.
49. 2×10^{-6}.
50. 4.81×5.96.
51. $(4.81)^3$.
52. $\dfrac{4.81}{5.96}$.
53. $\sqrt[3]{5.96}$.
54. $4.81 \times (5.96)^2$.

Using logarithms, find the value of each of the following:

55. $483 \times 2.656 \times 0.783$.
56. $413.6 \div 0.5718$.
57. $(40.21)^3 \times \sqrt{6.845}$.
58. $(61.63)^2 \div \sqrt[3]{0.0563}$.
59. $\dfrac{(3.92)^3 \times 72.16}{\sqrt[4]{654}}$.
60. $\dfrac{\sqrt[5]{594} \times 0.8614}{2.23^3}$.
61. $\dfrac{\sqrt{10.09} \times \sqrt[3]{258.7}}{1.06^8}$.
62. $\sqrt{\dfrac{15.92 \times 0.182}{0.00526 \times 196^5}}$.
63. $2600(1.025)^{14}$.
64. $\sqrt[5]{\dfrac{(6.153)^2 \times 3500}{52}}$.
65. $\sqrt[4]{\dfrac{1.26 \times (0.037)^3}{(1.89)^2}}$.

Write each of the following in the notation of logarithms:

66. $2^8 = 256$.
67. $27^{\frac{1}{3}} = 3$.
68. $10^{1.6021} = 40$.

Find the value of x in each of the following:

69. $\log_3 27 = x$.
70. $\log_{10} x = 4$.
71. $\log_x 32 = 5$.

72. If $\log \sqrt{x} = 0.7211$, what is $\log x^2$?

73. Given $\log x^2 = 0.6290$; find $\log 10x$.

74. Find the value of $\log 7132 - \log 7.132$.

75. The formula for the volume of a circular cylinder is $V = \pi R^2 H$; find by the use of logarithms the value of R when $V = 906.0$, $H = 14.60$, and $\pi = 3.142$.

76. Using the formula $A = P\left(1 + \frac{r}{2}\right)^{2n}$, find the amount of \$6990 at 4% interest compounded semiannually for 7 yr.

77. Find the interest earned by \$457 invested at 6% for 10 yr. if the interest is compounded semiannually.

78. Using the formula $A = P(1 + r)^n$, find the principal which will amount in 8 yr. to \$11,290 if the interest is compounded annually at 4%.

II. The Equation, the Formula, and the Graph

1. The Equation

Solve each of the following equations, and check each solution:

1. $15(x - 1) - (x - 3)^2 = 30 - (x + 15)(x - 3)$.
2. $(2x + \frac{1}{2})(x - \frac{1}{3}) = (x + \frac{1}{2})(2x + \frac{1}{3}) + \frac{1}{6}$.
3. $b(a + x)(b + x) - ab(b + c) = a^2c + bx^2$.
4. $\frac{x}{2} - 2\left(\frac{4x}{5} - 3\right) = 4 - \frac{3}{2}\left(\frac{x}{2} + 1\right)$.
5. $\frac{5(w - 4)}{2w} - \frac{2}{3}\left(6 - \frac{5w + 8}{w}\right) = \frac{2}{3}$.
6. $\frac{n}{1 - n} - \frac{1}{n - 1} = \frac{n^2 - 3}{1 - n^2} - \frac{4}{n + 1}$.
7. $3x + 5 : x^2 - 1 = 1 : x + 1$.

Solve the following sets of equations, and check each set of roots:

8. $5x - 3y = 35$,
   $3x + 4y = -8$.
9. $1.2x - 0.8y = 6.8$,
   $0.5x + 3.2y = 4.6$.
10. $\frac{x}{7} + \frac{y}{6} = -3$,
    $\frac{x}{5} + \frac{y}{2} = 7$.
11. $5a - b = 2c + 13$,
    $3a = 4c$,
    $\frac{3a - 2c}{4} - b = \frac{1}{2}$.
12. $2x = 8 - y$,
    $5x + 7y = 14 + z$,
    $\frac{3x + 2y}{2} = \frac{5}{2} - z$.

Solve the following quadratic equations, and give all irrational roots correct to the nearest 0.01:

13. $\frac{x - 1}{x + 1} = \frac{5}{2} + \frac{x + 1}{1 - x}$.
14. $\frac{x - 5}{x + 3} - \frac{1}{2} = \frac{80}{x^2 - 9} + \frac{x - 8}{3 - x}$.

15. $3x^2 + x - 10 = 0$.
16. $x^2 = 4x + 51$.
17. $5x^2 - 15x + 11 = 0$.
18. $0.3x^2 - 4x = 1.01$.

Without solving the following equations, state the nature of the roots in each case and also give the sum and product of the roots:

19. $2y^2 - 5y = 3$.
20. $9x - 7 = 3x^2$.
21. $2x^2 - 4x - 7 = 0$.
22. $x^2 - 5x = x - 16$.

State for what values of k the roots of each of the following equations are (1) real and equal, (2) real and unequal, (3) imaginary:

23. $x^2 + 2x + k = 0$.
24. $3x^2 + 7x + k = 0$.
25. $kx^2 + 4x = -1$.
26. $9x^2 = -kx - 16$.

In each of the following cases form the equations whose roots have the given values:

27. $2\frac{1}{2}, -3$.
28. $5 \pm \sqrt{2}$.
29. $-3 \pm \sqrt{7}$.
30. $2 \pm \sqrt{6}$.

31. What must be the value of m in order that one root of $2x^2 - mx - 5 = 0$ shall be -5?

32. In the equation $x^2 = 5x - p$, the difference between the roots is 11. Find the value of p.

Solve each of the following equations and check each result:

33. $3\sqrt{2x} - 5 = 0$.
34. $5 - \sqrt{x^2 - 5} = x$.
35. $\sqrt{1+x} + \sqrt{x} = \dfrac{2}{\sqrt{1+x}}$.
36. $2\sqrt{x} - \sqrt{4x-3} = \dfrac{1}{\sqrt{4x-3}}$.
37. $3x + 7 - 2\sqrt{3x+7} = 15$.
38. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 2 = 0$.
39. $x^{-4} - 10x^{-2} = -9$.
40. $2\sqrt{x} + 2x^{-\frac{1}{2}} = 5$.
41. $(x^2 - 3x)^2 - 8(x^2 - 3x) = 20$.

Solve each of the following equations for x:

42. $x^{\frac{3}{4}} = -8$.
43. $8^{3x-2} = 16^{2x-3}$.
44. $9^{\frac{2}{3}x} = 27^{-1}$.
45. $5^x = 22$.
46. $3^{x+2} = 405$.
47. $\log_{2.718} 16 = x$.

Solve each of the following pairs of equations:

48. $x^2 = 4 - y$,
$4x = 1 - 4y$.

49. $5x^2 - 3xy = 14$,
$4x - y = 7$.

50. $2x^2 - 5x - 2y = 2$,
$3x = 2y - 22$.

51. $8y - 9x = -xy$,
$x - 3y = -5$.

52. $x^2 + y^2 = 29$,
$xy - 10 = 0$.

53. $3x^2 - 4y^2 = 32$,
$2x^2 + 3y^2 = 44$.

54. $x^3 - 8y^3 = -37$,
$x - 2y = -1$.

55. $x^2 + 6xy = 28$,
$8y^2 + xy - 4 = 0$.

2. The Formula

Solve the following formulas for the letters indicated:

1. $F = \frac{W}{2R}(R - r)$ for W; for R.

2. $L = n\sqrt{c^2 + l^2}$ for n; for c.

3. $h = \frac{wv}{1 - 25r}$ for v; for r.

4. $T = 2\pi rh + 2\pi r^2$ for r.

5. $t^2 = dh + h^2$ for h.

6. $L = 2C + 3\frac{3}{8}(R + r)$ for C.

7. $s = vt + \frac{1}{2}gt^2$ for t.

8. $h = r - \sqrt{r^2 - (\frac{1}{2}w)^2}$ for w.

9. $T = 2\pi\sqrt{\frac{K^2}{gh}}$ for K.

10. $A = \frac{0.39Q}{\sqrt{W}}$ for W.

11. The curved surface of a circular cone whose altitude is h and the radius of whose base is r may be found by the formula $S = \pi r\sqrt{r^2 + h^2}$. Find S to the nearest tenth when $r = 8$, $h = 10$, and $\pi = 3.14$.

12. The formula for the volume (V) of the round head of a screw is $V = \pi h\left(\frac{D^2}{8} + \frac{h^2}{6}\right)$, where D is the diameter of the head and h is the height of the head. Find V when $D = 0.75$ in. and $h = 0.28$ in. (Use 3.14 as the value of π.)

13. The distance passed over in t seconds by a ball starting from rest and rolling down a smooth plane inclined at an angle

of 30° with the horizontal is given by the formula $s = \frac{1}{4}\, gt^2$. Find s when $g = 32.2$ and $t = 3$.

14. The horizontal distance (range) traversed by a body projected with a velocity of v feet per second at an angle of 45° with the horizontal is given by the formula $R = \frac{v^2}{g}$. Find R when $v = 250$ and $g = 32.2$.

15. The weight of air in pounds per cubic foot is given by the formula $W = \frac{1.325\ P}{460 + T}$, where P is the barometric pressure in inches of mercury and T is the temperature in degrees Fahrenheit. How does W change (1) if P increases and T remains constant? (2) if T increases and P remains constant?

3. The Graph

Solve graphically each of the following pairs of equations. If the roots are not integral, give results to the nearest tenth.

1. $x + 2\,y = 10$, $3\,y + 6 = 2\,x$.
3. $x^2 = -2\,y + 8$, $y = x$.
5. $2\,x - y^2 = 4$, $y - 6 = 2\,x$.
2. $x^2 + y^2 = 50$, $4\,x + 7\,y = 21$.
4. $xy = 12$, $3\,x - 5\,y = 6$.
6. $x^2 + 4\,y^2 = 32$, $x^2 - y^2 = 12$.

Plot the graph of the function $x^2 - 2\,x - 4$ for integral values of x from -2 to 4 inclusive. From the graph determine the roots of the following equations, giving results to the nearest tenth in the case of irrational roots.

7. $x^2 - 2\,x - 4 = -1$.
8. $x^2 - 2\,x - 4 = 0$.
9. $x^2 - 2\,x = 8$.
10. $x^2 - 2\,x + 5 = 4$.

In the case of each of the following equations determine by inspection the nature of the curve you would obtain if you plotted the graph of the equation:

11. $4\,x^2 + y^2 = 16$.
12. $2\,x^2 + 2\,y^2 = 8$.
13. $x^2 - y = 9$.
14. $5\,xy = -40$.
15. $y^2 + x = 4\,y$.
16. $x^2 - 3\,y^2 = 12$.

17. Write the equation of a circle whose centre is the origin and whose radius is 6.

18. Write the equation of a hyperbola symmetric with respect to both axes and having the intercepts $x = \pm 3$.

19. Given the function $x^2 - 6x - 7$. Draw the graph of the function and from it determine (1) the roots of the equation $x^2 - 6x - 7 = 0$ and (2) the y-intercept.

20. Write the equation of the ellipse with centre at the origin, major axis 5, and minor axis 3. Explain why there are two solutions.

21. Express the area of a rectangular field which can be inclosed with 144 rd. of fence as a function of the length (x rd.). Illustrate graphically. What is the greatest area possible?

III. Problems

1. Two numbers are in the ratio of 7 : 9; but if 14 is added to each, they will be in the ratio of 5 : 6. Find the numbers.

2. A rectangular flower bed is 6 ft. by 12 ft. A gardener wishes to double the size by spading around it a strip of uniform width. Find to the nearest 0.1 ft. the width of the strip.

3. A dealer paid \$72 for a radio set. At what price should he mark it so that he may sell it at a discount of 10 % of the marked price and still make a profit of 20 % of the sale price?

4. How long will it take a motorist travelling at an average rate of 40 mi. an hour to overtake another motorist who started from the same place $2\frac{1}{2}$ hr. earlier and is travelling at an average rate of 25 mi. an hour?

5. A 5 % solution of salt, weighing 160 lb., is to be reduced to a 4 % solution. How many pounds of water must be added?

6. If x is a positive number, arrange the ratios $\frac{3}{4}$, $\frac{3+x}{4+x}$ and $\frac{3-x}{4+x}$ according to values.

7. A man walked into the country at an average rate of 4 mi. an hour and returned over the same route, riding in a trolley car which averaged 16 mi. per hour. He made the round trip in 4 hr. How long was he riding on the trolley?

8. An open box is made from a square piece of tin by cutting out a 6-inch square from each corner and turning up the sides. The box contains 150 cu. in. Find the area of the original square.

9. A man made two investments totalling $6300. On the first he received $5\frac{1}{2}\%$ interest and on the second 4%. The annual income from the first investment was $14 more than from the second investment. How much money did he invest at each rate?

10. It cost $144 to manufacture a certain bedroom set. At what price should the manufacturer mark it so that he can sell it at a discount of 10% on the marked price and still make a profit of 20% on the sale price?

11. The speed of an airplane is 90 mi. per hour in calm weather. Flying with the wind it can cover a certain distance in 4 hr., but when flying against the wind it can cover only $\frac{3}{5}$ of this distance in the same time. What is the velocity of the wind?

12. A ladder 26 ft. long is leaning against a house. Its foot is 7 ft. from the house. How far above the ground is the top of the ladder?

13. A man bought a lot and built a house upon it as an investment. The total cost was $10,000. For how much must he sell the property in order to make 14% on his investment, after paying 5% commission to a real-estate agent for selling it?

14. If three times the larger of two numbers is divided by the smaller, the quotient is 6 and the remainder is 6. If five times the smaller is divided by the larger, the quotient is 2 and the remainder is 3. Find the numbers.

15. Two motorists travel around a circular track in 6 min. and 8 min. respectively. If they start together and travel in the same direction, after how many minutes will they be together again?

16. The two legs of a right triangle are 3 in. and 4 in. respectively. Find by how many inches the longer leg must be extended in order that the hypotenuse may become 2 in. longer.

17. An electric current (C) varies inversely as the resistance (R). If $C = 6$ when $R = 30$, find C when $R = 50$.

18. A tourist finds that he can cover m miles in h hours. If he plans to travel at that rate for d hours each day, how many days will it take him to cover 1000 mi.?

19. A sum of \$820, when put at simple interest for a certain number of years at a given rate, would amount to \$955.30. If the time were 1 yr. less and the rate $\frac{1}{2}\%$ more, the amount would be \$918.40. Find the time and the rate.

20. The number of gallons of gasoline which can be stored in a cylindrical tank is a constant times the product of the height and the square of the radius. A certain tank 12 ft. in height holds 1000 gallons.

(1) How many gallons can be stored in a tank having the same diameter and 15 ft. high?

(2) How many gallons can be stored in a tank 15 ft. high and having $1\frac{1}{2}$ times the diameter of the first tank?

3. A Final Test. The test which follows may be regarded as a comprehensive review of the principal objectives around which this Second Course in Algebra was organized.

Comprehensive Mastery Test

Part I. Algebraic Technique

1. Give an illustration of each of the following:

(1) A linear function.
(2) A trinomial square.
(3) An irrational number.
(4) An imaginary number.
(5) A quadratic function.
(6) A complex fraction.
(7) An exponential equation.
(8) A logarithmic function.

If $A = 6x^2 - x - 7$, and $B = 2x - 3$, *find*

2. $A + B$. 3. $A - B^2$. 4. $A \times B$. 5. $A \div B$.

Factor completely each of the following expressions:

6. $c^4 - 81d^4$.

7. $27 + 8x^3$.

8. $x^3 - 5x - 2$.

9. $3x^{3a} - 5x^{2a} - 2x^a$.

10. $4x^2 - 2x + 0.25$.

11. $x^2 - 2xy + y^2 - a^2$.

12. If $3x - 2$ is a factor of $kx^2 + 7x - 10$, find the value of k.

Perform the indicated operations in each case and express the result as a fraction in its lowest terms:

13. $\dfrac{3x-4}{x-1} + \dfrac{2x-3}{1-x}$.

14. $\dfrac{2a+3}{a^2-4} - 2\left(\dfrac{1}{8a+16}\right)$.

15. $\left(\dfrac{x}{3} - \dfrac{3}{x}\right) \div \left(\dfrac{2x-6}{3x}\right)$.

16. $\dfrac{a^{\frac{3}{4}}}{a^{-\frac{1}{4}} + a^{-\frac{5}{4}}}$.

17. $\dfrac{x - \dfrac{9}{x}}{\dfrac{6}{x^2} + \dfrac{1}{x} - 1} \times \left(\dfrac{2-x^2}{3x+x^2} + 1\right)$.

18. The national wealth of Canada in a recent year was estimated at 31,000,000,000 dollars. Express this numerical value in exponential form.

19. Express 365×10^{-4} as a decimal fraction.

20. Show by substitution that 9 is a root of the equation

$$x^{\frac{1}{2}} + 6x^{-\frac{1}{2}} = 5x^0.$$

21. Simplify $2\sqrt{27} - 18\sqrt{\frac{1}{3}} + 3\sqrt{\frac{1}{3}}$.

22. What is the value of $x^2 - 2x - 3$ when $x = \sqrt{2} - 1$?

23. If $x = \sqrt{3}$, show that $\dfrac{x}{x-1} - \dfrac{3}{x} = \dfrac{3-\sqrt{3}}{2}$.

24. Rationalize the denominator of $\dfrac{1}{\sqrt{7}+2}$ and find the value of the result to the nearest hundredth.

Find the value of each of the following, if $x = 9$:

25. $6\sqrt{\frac{4}{x^2}}$. 26. $\sqrt[3]{\frac{-3}{x}}$. 27. $4\,x^{-\frac{3}{2}}$. 28. $x^{-\frac{3}{4}}$.

In the case of each of the following statements indicate whether it is true or false. If it is false, write it in correct form.

29. $(a-b)^2 = a^2 - b^2$.

30. If $x = 3$, $2\,x^2 = 18$.

31. $\frac{n_1}{a} + \frac{n_2}{b} = \frac{n_1 + n_2}{ab}$.

32. $-\frac{x-3}{2-x} = -\frac{x-3}{x-2}$.

33. $\sqrt{2} + \sqrt{8} = \sqrt{10}$.

34. $\sqrt{a+b-2\sqrt{ab}} = \sqrt{a} - \sqrt{b}$.

35. $\frac{\log x}{\log y} = \log x - \log y$.

36. $\log \sqrt[3]{2\,n} = \frac{2 \log n}{3}$.

37. Find the value of $\log 6508 - \log 65.08$.

38. How many figures or digits are there in the number corresponding to 3^{50}? Find the approximate answer by logarithms.

39. Find the value of $\frac{\sqrt[3]{493} \times 0.7235}{4.36^2}$.

40. Using the relation $A : A' = s_1^2 : s_2^2$, find by logarithms the value of s_1 if $A = 1600$, $A' = 7290$, and $s_2 = 310.6$.

Part II. Equations, Formulas, Graphs

Solve each of the following equations for x:

41. $2\,x(x+2) - (5\,x-1) = (x+2)^2 - 7$.

42. $2\,x^2 + x = 6$.

43. $\frac{x}{a} - \frac{3(x-bc)}{b} = 7\,c$.

44. $12 = x + \sqrt{x^2 - 24}$.

45. $\frac{x^2}{4\,m} + \frac{x}{2} + k = 0$.

Solve each of the following sets of equations:

46. $5\,x - y = 2$,
$3\,x - 2\,y = -10$.

47. $3\,x + 5\,y - 2\,z = 5$,
$2\,x - y + 8\,z = 9$,
$4\,x + 3\,y + z = 3$.

48. $5\,x^2 - 3\,xy = 14$,
$4\,x - y = 7$.

49. $x^2 - y^2 = 21$,
$x + 2\,y^2 = 13$.

50. Find to the nearest hundredth the roots of $3x^2-5x-4=0$.

51. Given the equation $4x^2 - 3x - 1 = 0$. (1) Find the value of the discriminant. (2) What is the nature of the roots? (3) What is the ratio of the sum of the roots to their product?

52. Given the equation $3x^2 - 4x + m = 0$. Determine the value of m for which (1) one root is 0; (2) one root exceeds the other by 2; (3) the roots are real and equal; (4) the roots are imaginary; (5) one root is three times the other.

53. Form the equation whose roots have the following values:

(1) $\frac{1}{3}$ and $\frac{1}{6}$.
(2) $\sqrt{2}$ and $-\sqrt{2}$.
(3) $\sqrt{3}-\sqrt{2}$ and $\sqrt{3}+\sqrt{2}$.
(4) $2+\sqrt{3}$ and $2-\sqrt{3}$.
(5) $0, 1, 3\pm\sqrt{2}$.

54. Given the linear function $y = mx + b$. If $y = -1$ when $x = 1$, and $y = 5$ when $x = 4$, find the values of m and b.

In the case of each of the following equations, determine by inspection the nature of the graph which corresponds to the equation:

55. $y = 2x^2$.
56. $y = 3x - 2$.
57. $x^2 - y^2 = 9$.
58. $x = y^2 - 4y + 4$.
59. $x^2 + y^2 = 4$.
60. $4x^2 + y^2 = 16$.
61. $x^2 + 9y^2 = 36$.
62. $xy = 6$.

63. In the case of the graph of each of the following equations, determine the slope and the y-intercept:

(1) $y = 2x$.
(2) $y = \frac{2}{3}x$.
(3) $y = x + 7$.
(4) $y = \frac{1}{2}x + 1$.
(5) $y = 3x - 2$.
(6) $y = -\frac{3}{4}x - 5$.
(7) $2x - 3y = 6$.
(8) $3x - 5y = 21$.
(9) $4x + y = -3$.

64. What must be the value of a if the graph of $3x - 5y = a$ passes through the point whose coördinates are $x = 3$ and $y = 1$?

Solve graphically each of the following pairs of equations:

65. $3x + 2y = 3$,
$y - 3x = 10\frac{1}{2}$.

66. $x^2 + y^2 = 25$,
$x - y = 1$.

67. Solve for p the formula $\frac{R-p}{q-R}=\frac{p}{q}$.

68. If $x=\frac{1}{y}-n$, find the value of $(x+n)^2$ in terms of y.

69. Solve for n the formula $A=P(1+r)^n$. (Use logarithms.)

70. If y varies inversely as x, and if $y=25$ when $x=4$, find the value of y when $x=20$.

71. The dimensions of a rectangular box are b, d, and h. What is the effect on the volume (V) if b and d are multiplied by n?

72. If $R=kS^4$, what change is produced in R if S is doubled?

73. Using the formula $A=\frac{s}{2}\sqrt{\left(r-\frac{s}{2}\right)\left(r+\frac{s}{2}\right)}$, find the value of A when $r=10$ and $s=12$.

74. From the table below, find the formula connecting p and q:

$p=$	10	14	20	25	30	34	40
$q=$	2	4.8	9	12.5	16	18.8	23

75. A ball started to roll down an inclined plane with a certain initial velocity. The successive distances d, in inches, which it covered during a time interval t, in seconds, from 1 to 6, are given in the following table. Find the formula connecting d and t.

$t=$	1	2	3	4	5	6
$d=$	10	36	78	136	210	300

Part III. Problems

76. A man invested \$11,700 in two enterprises. At the end of the first year he found that he had gained 6% on one of the sums invested and lost 4% on the other. His net profit for the year was \$282. How much did he invest in each enterprise?

77. The members of the Senior Class of a certain high school were to be assessed equally to buy a class gift costing \$150. During the term, 20 pupils dropped out of school, and as a result the assessment for each of those remaining had to be raised \$0.25. How many members did the class have originally?

78. One printing press can do a piece of work in 5 hr., while another press requires 8 hr. for the same job. After the first press has been running for 2 hr. on the job, it is stopped and the work is transferred to the second press. How long must the second press run to complete the job?

79. A number of men each subscribed the same amount to make up a deficit of \$500, but 5 of the men failed to pay and thus increased the share of each of the others by \$5. How much did each pay, and how many men subscribed originally?

80. The area of a rectangle is 240 sq. ft. and its dimensions are in the ratio 5 : 3. Find the longer dimension.

81. How many quarts of maple syrup 45% pure must be mixed with 10 qt. that are 25% pure to make a syrup that will be 40% pure?

82. Mr. Brown took an automobile trip, leaving his town at a certain hour and travelling at a uniform rate. An hour later Mr. Clark started out from the same place and travelled in the same direction at an hourly rate 5 mi. greater than that of Mr. Brown. After travelling 100 mi. he overtook Mr. Brown. Find the rate of each man.

83. Find two numbers in the ratio $a : b$ whose sum is s.

84. At 4% interest, compounded annually, what is the amount of an investment of \$2000 at the end of 10 yr.?

85. Two pulleys having unequal diameters are connected by a belt. Their diameters are d_1 and d_2 respectively. The number of revolutions made by the first pulley during a certain period of time is n_1, while that of the second pulley is n_2. Show that $n_1 : n_2 = d_2 : d_1$.

TABLES

TABLE OF SQUARE ROOTS OF NUMBERS FROM 1 TO 209

n	—	1	2	3	4	5	6	7	8	9
—		1.000	1.414	1.732	2.000	2.236	2.449	2.646	2.828	3.000
10	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
20	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
30	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
40	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
50	7.071	7.141	7.211	7.280	7.348	7.416	7.483	7.550	7.616	7.681
60	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
70	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
80	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
90	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.899	9.950
100	10.000	10.050	10.100	10.149	10.198	10.247	10.296	10.344	10.392	10.440
110	10.488	10.536	10.583	10.630	10.677	10.724	10.770	10.817	10.863	10.909
120	10.954	11.000	11.045	11.091	11.136	11.180	11.225	11.269	11.314	11.358
130	11.402	11.446	11.489	11.533	11.576	11.619	11.662	11.705	11.747	11.790
140	11.832	11.874	11.916	11.958	12.000	12.042	12.083	12.124	12.166	12.207
150	12.247	12.288	12.329	12.369	12.410	12.450	12.490	12.530	12.570	12.610
160	12.649	12.689	12.728	12.767	12.806	12.845	12.884	12.923	12.961	13.000
170	13.038	13.077	13.115	13.153	13.191	13.229	13.266	13.304	13.342	13.379
180	13.416	13.454	13.491	13.528	13.565	13.601	13.638	13.675	13.711	13.748
190	13.784	13.820	13.856	13.892	13.928	13.964	14.000	14.036	14.071	14.107
200	14.142	14.177	14.213	14.248	14.283	14.318	14.353	14.387	14.422	14.457

TABLE OF CUBE ROOTS OF NUMBERS FROM 1 TO 109

n	—	1	2	3	4	5	6	7	8	9
—		1.000	1.260	1.442	1.587	1.710	1.817	1.913	2.000	2.080
10	2.154	2.224	2.289	2.351	2.410	2.466	2.520	2.571	2.621	2.668
20	2.714	2.759	2.802	2.844	2.884	2.924	2.962	3.000	3.037	3.072
30	3.107	3.141	3.175	3.208	3.240	3.271	3.302	3.332	3.362	3.391
40	3.420	3.448	3.476	3.503	3.530	3.557	3.583	3.609	3.634	3.659
50	3.684	3.708	3.733	3.756	3.780	3.803	3.826	3.849	3.871	3.893
60	3.915	3.936	3.958	3.979	4.000	4.021	4.041	4.062	4.082	4.102
70	4.121	4.141	4.160	4.179	4.198	4.217	4.236	4.254	4.273	4.291
80	4.309	4.327	4.344	4.362	4.380	4.397	4.414	4.431	4.448	4.465
90	4.481	4.498	4.514	4.531	4.547	4.563	4.579	4.595	4.610	4.626
100	4.642	4.657	4.672	4.688	4.703	4.718	4.733	4.747	4.762	4.777

TABLE OF MANTISSAS

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

TABLE OF MANTISSAS

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
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